Sums of Squares Relaxation of Polynomial Optimization Problems

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- **An introduction to the recent development of SOS relaxation for computing global optimal solutions of POPs.**

- **Exploiting sparsity in SOS relaxation to solve large scale POPs.**

Outline

1. **POPs (Polynomial Optimization Problems)**
2. **Nonnegative polynomials and SOS (Sum of Squares) polynomials**
3. **SOS relaxation of unconstrained POPs**
4. **Conversion of SOS relaxation into an SDP (Semidefinite Program)**
5. **Exploiting structured sparsity**
6. **SOS relaxation of constrained POPs — very briefly**
7. **Numerical results**
8. **Concluding remarks**

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$\mathbb{R}^n$: the n-dim Euclidean space.

$x = (x_1, \ldots, x_n) \in \mathbb{R}^n$: a vector variable.

$f_j(x)$: a multivariate polynomial in $x \in \mathbb{R}^n$ $(j = 0, 1, \ldots, m)$.

POP: min $f_0(x)$ sub.to $f_j(x) \geq 0$ $(j = 1, \ldots, m)$.

Example: $n = 3$

$$\begin{array}{rl} \min & f_0(x) \equiv x_1^3 - 2x_1x_2^2 + x_1^2x_2x_3 - 4x_3^2 \\ \text{sub.to} & f_1(x) \equiv -x_1^2 + 5x_2x_3 + 1 \geq 0, \\ & f_2(x) \equiv x_1^2 - 3x_1x_2x_3 + 2x_3 + 2 \geq 0, \\ & f_3(x) \equiv -x_1^2 - x_2^2 - x_3^2 + 1 \geq 0, \\ & x_1(x_1 - 1) = 0 \text{ (0-1 integer)}, \\ & x_2 \geq 0,\ x_3 \geq 0,\ x_2x_3 = 0 \text{ (complementarity)}. \end{array}$$

- Various problems can be described as POPs.
- A unified theoretical model for global optimization in nonlinear and combinatorial optimization problems.

POP: min $f_0(x)$ sub.to $f_j(x) \geq 0$ $(j = 1, \ldots, m)$.

[1] **J.B.Lasserre, "Global optimization with polynomials and the problems of moments", *SIAM J. on Optim.* (2001).**

[2] **P.A.Parrilo, "Semidefinite programming relaxations for semialgebraic problems". *Math. Prog.* (2003).**

- **[1] $\Longrightarrow$ SDP relaxation — primal approach.**
- **[2] $\Longrightarrow$ SOS relaxation — dual approach.**
- **[1] and [2] are dual to each other.**

(a) Lower bounds for the optimal value.

(b) Convergence to global optimal solutions in theory.

(c) Large-scale SDPs require enormous computation.

(d) "Exploit structured sparsity" to solve large scale POPs.

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$f(x)$: a nonnegative polynomial $\Leftrightarrow$ $f(x) \geq 0$ $(\forall x \in \mathbb{R}^n)$.
$\mathcal{N}$: the set of nonnegative polynomials in $x \in \mathbb{R}^n$.

$f(x)$: an SOS (Sum of Squares) polynomial

$$\Updownarrow$$

$$\exists \text{ polynomials } g_1(x), \ldots, g_k(x);\ f(x) = \sum_{i=1}^{k} g_i(x)^2.$$

$\mathcal{SOS}_*$: the set of SOS. Obviously, $\mathcal{SOS}_* \subset \mathcal{N}$.
$\mathcal{SOS}_{2r} = \{f \in \mathcal{SOS}_* : \deg f \leq 2r\}$: SOSs with degree ar most $2r$.

$n = 2$. $f(x_1, x_2) = (x_1^2 - 2x_2 + 1)^2 + (3x_1x_2 + x_2 - 4)^2 \in \mathcal{SOS}_4$.

- In theory, $\mathcal{SOS}_*$ (SOS) $\subset \mathcal{N}$. $\mathcal{SOS}_* \neq \mathcal{N}$ in general.
- In practice, $f(x) \in \mathcal{N} \backslash \mathcal{SOS}_*$ is rare.
- So we replace $\mathcal{N}$ by $\mathcal{SOS}_* \Longrightarrow$ SOS Relaxations.

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$$\mathcal{P}: \quad \min_{x \in \mathbb{R}^n} \; f(x), \text{ where } f \text{ is a polynomial with } \deg f = 2r$$

$\mathcal{P}$: $\min_{x \in \mathbb{R}^n} f(x)$, where f is a polynomial with $\deg f = 2r$

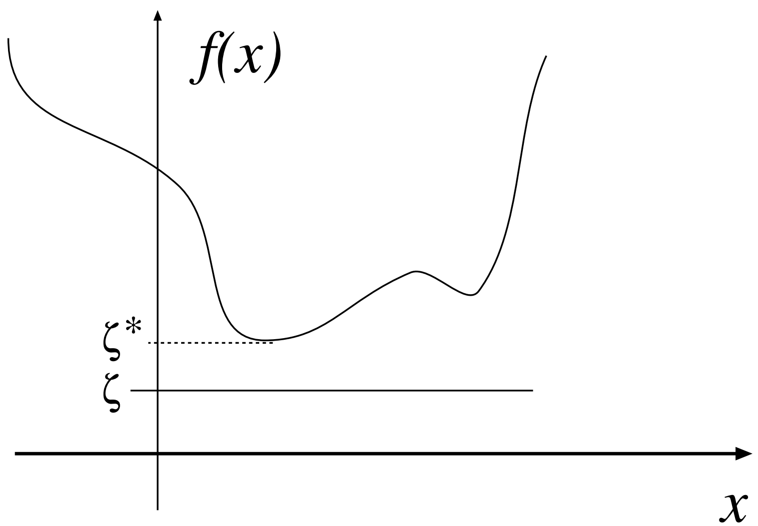
$$\Updownarrow$$

$\mathcal{P}$': max ζ s.t $\quad f(x) - \zeta \geq 0 \; (\forall x \in \mathbb{R}^n)$

$$\Updownarrow$$

$f(x) - \zeta \in \mathcal{N}$ (the nonnegative polynomials)

Here x is an index describing inequality constraints.



$\mathcal{P}$: $\min_{x \in \mathbb{R}^n} f(x)$, where f is a polynomial with deg $f = 2r$

$$\Updownarrow$$

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$$\Updownarrow$$

$f(x) - \zeta \in \mathcal{N}$ (the nonnegative polynomials)

Here x is an index describing inequality constraints.

$\Sigma \subset \mathrm{SOS}_{2r} \subset \mathrm{SOS}_* \subset \mathcal{N}$ $\Downarrow$ a subproblem of $\mathcal{P}'$ = a relaxation of $\mathcal{P}$

$\mathcal{P}$": max ζ sub.to $f(x) - \zeta \in \Sigma$

SOS_* (SOS_{2r} =) the set of SOS polynomials (with degree $\leq 2r$).

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SOS_* (SOS_{2r} =) the set of SOS polynomials (with degree $\leq 2r$).

- the min.val of $\mathcal{P}$ = the max.val of $\mathcal{P}'$ $\geq$ the max.val of $\mathcal{P}$".
- $\mathcal{P}$" can be solved as an SDP (Semidefinite Program) — next.
- In practice, we can exploit structured sparsity of the Hessian matrix of f to reduce the size of Σ — later.

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Conversion of SOS relaxation into an SDP — 1

What is an SDP (Semidefinite Program)?

- **An extension of LP (Linear Program) in $\mathbb{R}^n$ to the space $\mathcal{S}^n$ of symmetric matrices;**

  variable **a vector** $\boldsymbol{x} \in \mathbb{R}^n \Longrightarrow \boldsymbol{X} \in \mathcal{S}^n$.

  inequality $\mathbb{R}^n \ni \boldsymbol{x} \geq \mathbf{0} \Longrightarrow \mathcal{S}^n \ni \boldsymbol{X} \succeq \boldsymbol{O}$ **(positive semidefinite).**

- **Can be solved by the interior-point method.**
- **Lots of applications.**

Conversion of SOS relaxation into an SDP — 2

$a_p \in \mathbb{R}^n$ $(p = 0, 1, 2, \ldots, m)$, $b_p \in \mathbb{R}$ $(p = 1, 2, \ldots, m)$: data.
$x \in \mathbb{R}^n$: variable.
$a_p \cdot x = \sum_{j=1}^n [a_p]_j x_j$ (the inner product).

LP (Linear Program):

$$\begin{array}{ll} \max & a_0 \cdot x \\ \text{s.t.} & a_p \cdot x = b_p \ (p = 1, \ldots, m), \ x \geq 0. \end{array}$$

SDP (Semidefinite Program):

$$\begin{array}{ll} \max & A_0 \bullet X \\ \text{s.t.} & A_p \bullet X = b_p \ (p = 1, \ldots, m), \ X \succeq O. \end{array}$$

$A_p \in \mathcal{S}^n$ $(p = 0, 1, 2, \ldots, m)$, $b_p \in \mathbb{R}$ $(p = 1, 2, \ldots, m)$: data
$X \in \mathcal{S}^n$: variable.
$A_p \bullet X = \sum_{i=1}^n \sum_{j=1}^n [A_p]_{ij} X_{ij}$ (the inner product).

$\mathcal{S}^n$: the set of $n \times n$ real symmetric matrices.
$X \succeq O$: $X \in \mathcal{S}^n$ is positive semidefinite.

Conversion of SOS relaxation into an SDP — 3

Representation of

$$\mathcal{SOS}_{2r} \equiv \left\{ \sum_{j=1}^{k} g_j(x)^2 : \exists k \geq 1,\ g_j(x) : \text{ degree at most } r \right\} \subset \mathcal{SOS}_*.$$

$\forall$ r-degree poly. $g(x)$ $\exists a \in \mathbb{R}^{d(r)}$; $g(x) = a^T u_r(x)$, where

$$u_r(x) = (1, x_1, x_2, \ldots, x_n, x_1^2, x_1x_2, x_1x_3, \ldots, x_n^2, \ldots, x_1^r, \ldots, x_n^r)^T,$$

(a column vector of a basis of r-degree polynomial),

$$d(r) = \binom{n+r}{r} : \text{ the dimension of } u_r(x).$$

$\Downarrow$

$$\begin{aligned}
\mathcal{SOS}_{2r} &= \left\{ \sum_{j=1}^{k} \left(a_j^T u_r(x) \right)^2 : k \geq 1,\ a_j \in \mathbb{R}^{d(r)} \right\} \\
&= \left\{ u_r(x)^T \left(\sum_{j=1}^{k} a_j a_j^T \right) u_r(x) : k \geq 1,\ a_j \in \mathbb{R}^{d(r)} \right\} \\
&= \left\{ u_r(x)^T V u_r(x) : V \text{ is a positive semidefinite matrix} \right\}.
\end{aligned}$$

Conversion of SOS relaxation into an SDP — 4

Example. $n = 2$, SOS of at most deg.2 polynomials in $x=(x_1, x_2)$.

$$\mathrm{SOS}_4 \equiv \left\{ \sum_{i=1}^{k} g_i(x)^2 \ : \ k \geq 1, \ g_i(x) \ \text{ is at most deg.2 polynomial} \right\}$$

$$= \left\{ \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_1^2 \\ x_1 x_2 \\ x_2^2 \end{pmatrix}^T V \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_1^2 \\ x_1 x_2 \\ x_2^2 \end{pmatrix} \ : \ V \text{ is a } 6 \times 6 \text{ psd matrix} \right\}$$

Conversion of SOS relaxation into an SDP — 5

Example : $f(x) = -x_1 + 2x_2 + 3x_1^2 - 5x_1^2x_2^2 + 7x_2^4$

max ζ sub.to $f(x) - \zeta \in$ SOS$_4$ (SOS of at most deg. 2 polynomials)

$\Updownarrow$

max ζ

Sum of Squares

$$\text{s.t.} \quad f(x) - \zeta = \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_1^2 \\ x_1x_2 \\ x_2^2 \end{pmatrix}^T \begin{pmatrix} V_{11} & V_{12} & V_{13} & V_{14} & V_{15} & V_{16} \\ V_{12} & V_{22} & V_{23} & V_{24} & V_{25} & V_{26} \\ V_{13} & V_{23} & V_{33} & V_{34} & V_{35} & V_{36} \\ V_{14} & V_{24} & V_{34} & V_{44} & V_{45} & V_{46} \\ V_{15} & V_{25} & V_{35} & V_{45} & V_{55} & V_{56} \\ V_{16} & V_{26} & V_{36} & V_{46} & V_{56} & V_{66} \end{pmatrix} \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_1^2 \\ x_1x_2 \\ x_2^2 \end{pmatrix}$$

$(\forall (x_1, x_2)^T \in \mathbb{R}^n), \quad 6 \times 6 \ V \succeq O$

$\Updownarrow$ Compare the coef. of $1,\ x_1,\ x_2,\ x_1^2,\ x_1x_2, x_2^2$ on both side of $=$

SDP (Semidefinite Program)

$$\max\ \zeta \ \text{s.t.} \quad -\zeta = V_{11},\ -1 = 2V_{12},\ 2 = 2V_{13},\ 3 = 2V_{14} + V_{22},$$
$$-5 = 2V_{46} + V_{55},\ 7 = V_{66},\ \text{all others}\ \ 0 = \cdots\ ,\ V \succeq O$$

In general, each equality constraint is a linear equation in ζ and V.

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$\mathcal{P}$: $\min_{x \in \mathbb{R}^n} f(x)$, where f is a polynomial with deg $f = 2r$

H : the sparsity pattern of the Hessian matrix of $f(x)$

$$H_{ij} = \begin{cases} \star \text{ if } i = j \text{ or } \partial^2 f(x)/\partial x_i \partial x_j \not\equiv 0, \\ 0 \text{ otherwise.} \end{cases}$$

$f(x)$: correlatively sparse $\Leftrightarrow$ $\exists$ a sparse Cholesky fact. of H.

(a) A sparse Chol. fact. is characterized as a sparse (chordal) graph $G(N, E)$; $N = \{1, \ldots, n\}$ and

$$E = \{(i, j) : H_{ij} = \star\} + \text{ "fill-in".}$$

(b) Let $C_1, C_2, \ldots, C_q \subset N$ be the maximal cliques of $G(N, E)$.

Sparse SOS relaxation

max ζ
s.t. $f(x) - \zeta \in \sum_{k=1}^{q}$ (SOS of polynomials in x_i $(i \in C_k)$)

Dense SOS relaxation

max ζ
s.t. $f(x) - \zeta \in$ (SOS of polynomials in x_i $(i \in N)$)

- Sparse relaxation is weaker but less expensive in practice.

Example: Generalized Rosenbrock function.

$$f(x) = \sum_{i=2}^{n} \left(100(x_i - x_{i-1}^2)^2 + (1 - x_i)^2\right).$$

Dense SOS relaxation
  max ζ
  s.t. $f(x) - \zeta \in$ (SOS of deg-2. poly. in $x_1, x_2, \ldots, x_n$)

- **The size of Dense grows very rapidly, so we can't apply Dense to the case $n \geq 20$ in practice.**

- **The Hessian matrix is sparse (tridiagonal).**
- **No fill-in in the Cholesky factorization.**
- $C_i = \{i-1, i\}$ $(i = 2, \ldots, n-1)$ **: the max. cliques.**

Sparse SOS relaxation
  max ζ
  s.t. $f(x) - \zeta \in \sum_{i=2}^{n}$(SOS of deg-2. poly. in x_{i-1}, x_i)

- **The size of Sparse grows linearly in n, and Sparse can process the case $n = 800$ in less than 10 sec.**

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POP: $\min\ f_0(x)$ sub.to $f_j(x) \geq 0\ (j = 1, \ldots, m)$.

- **Rough sketch of SOS relaxation of POP**

"Generalized Lagrangian Dual"

+

"SOS relaxation of unconstrained POPs"

$\Downarrow$

SOS relaxation of POP

- **Exploiting sparsity in SOS relaxation of POP**

POP: $\min\ f_0(x)$ sub.to $f_j(x) \geq 0\ (j = 1, \ldots, m)$.

Generalized Lagrangian function

$$L(x, \lambda_1, \ldots, \lambda_m) = f_0(x) - \sum_{j=1}^{m} \lambda_j(x) f_j(x)$$

for $\forall x \in \mathbb{R}^n$, $\forall \lambda_j \in \mathrm{SOS}_*$

If $\mathbb{R} \ni \lambda_j \geq 0$ then **L** is the standard Lagrangian function.

Generalized Lagrangian Dual

$$\max_{\lambda_1 \in \mathrm{SOS}_*, \ldots, \lambda_m \in \mathrm{SOS}_*} \ \min_{x \in \mathbb{R}^n} L(x, \lambda_1, \ldots, \lambda_m)$$

$$\Updownarrow$$

$$\begin{array}{ll} \max & \zeta \\ \text{s.t.} & L(x, \lambda_1, \ldots, \lambda_m) - \zeta \geq 0 \ (\forall x \in \mathbb{R}^n), \\ & \lambda_1 \in \mathrm{SOS}_*, \ldots, \lambda_m \in \mathrm{SOS}_* \end{array}$$

$\Downarrow$ SOS relaxation

$$\begin{array}{ll} \max & \zeta \\ \text{s.t.} & L(x, \lambda_1, \ldots, \lambda_m) - \zeta \in \Sigma_0 \\ & \lambda_1 \in \Sigma_1, \ldots, \lambda_m \in \Sigma_m. \end{array}$$

Here Σ_j denotes a set of SOS polynomials with a finite degree.

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Numerical results

Software

- **MATLAB for constructing sparse and dense SDP relaxation problems**
- **SeDuMi to solve SDPs.**

Hardware

- **2.4GHz Xeon cpu with 6.0GB memory.**

G.Rosenbrock function:

$$f(x) = \sum_{i=2}^{n} \left(100(x_i - x_{i-1}^2)^2 + (1 - x_i)^2\right)$$

- Two minimizers on $\mathbb{R}^n$: $x_1 = \pm 1$, $x_i = 1$ $(i \geq 2)$.
- Add $x_1 \geq 0 \Rightarrow$ a single minimizer.

		cpu in sec.	
n	ϵ_{obj}	Sparse	Dense
10	2.5e-08	0.2	10.6
15	6.5e-08	0.2	756.6
200	5.2e-07	2.2	—
400	2.5e-06	3.7	—
800	5.5e-06	6.8	—

$$\epsilon_{\text{obj}} = \frac{|\text{the lower bound for opt. value} - \text{the approx. opt. value}|}{\max\{1, |\text{the lower bound for opt. value}|\}}.$$

An optimal control problem from Coleman et al. 1995

$$\left.\begin{array}{l} \min \ \dfrac{1}{M} \displaystyle\sum_{i=1}^{M-1} \left(y_i^2 + x_i^2\right) \\ \text{s.t.} \ \ y_{i+1} = y_i + \dfrac{1}{M}(y_i^2 - x_i), \quad (i = 1, \ldots, M-1), \quad y_1 = 1. \end{array}\right\}$$

Numerical results on sparse relaxation

M	# of variables	ϵ_{obj}	ϵ_{feas}	cpu
600	1198	3.4e-08	2.2e-10	3.4
700	1398	2.5e-08	8.1e-10	3.3
800	1598	5.9e-08	1.6e-10	3.8
900	1798	1.4e-07	6.8e-10	4.5
1000	1998	6.3e-08	2.7e-10	5.0

$$\epsilon_{\text{obj}} = \frac{|\text{the lower bound for opt. value} - \text{the approx. opt. value}|}{\max\{1, |\text{the lower bound for opt. value}|\}},$$

ϵ_{feas} = the maximum error in the equality constraints,

cpu : cpu time in sec. to solve an SDP relaxation problem.

alkyl.gms : a benchmark problem from globallib

$$
\begin{array}{ll}
\text{min} & -6.3x_5x_8 + 5.04x_2 + 0.35x_3 + x_4 + 3.36x_6 \\
\text{sub.to} & -0.820x_2 + x_5 - 0.820x_6 = 0, \\
& 0.98x_4 - x_7(0.01x_5x_{10} + x_4) = 0, \\
& -x_2x_9 + 10x_3 + x_6 = 0, \\
& x_5x_{12} - x_2(1.12 + 0.132x_9 - 0.0067x_9^2) = 0, \\
& x_8x_{13} - 0.01x_9(1.098 - 0.038x_9) - 0.325x_7 = 0.574, \\
& x_{10}x_{14} + 22.2x_{11} = 35.82, \\
& x_1x_{11} - 3x_8 = -1.33, \\
& \text{lbd}_i \le x_i \le \text{ubd}_i \ (i = 1, 2, \ldots, 14).
\end{array}
$$

		Sparse			Dense (Lasserre)		
problem	n	ϵ_{obj}	ϵ_{feas}	cpu	ϵ_{obj}	ϵ_{feas}	cpu
alkyl	14	5.6e-10	2.0e-08	23.0	out of memory		

$$
\epsilon_{\text{obj}} = \frac{|\text{the lower bound for opt. value} - \text{the approx. opt. value}|}{\max\{1, |\text{the lower bound for opt. value}|\}},
$$

ϵ_{feas} = the maximum error in the equality constraints,

cpu : cpu time in sec. to solve an SDP relaxation problem.

Some other benchmark problems from globallib

		Sparse			Dense (Lasserre)		
problem	n	ϵ_{obj}	ϵ_{feas}	cpu	ϵ_{obj}	ϵ_{feas}	cpu
ex3_1_1	8	6.3e-09	4.7e-04	5.5	0.7e-08	2.5e-03	597.8
st_bpaf1b	10	3.8e-08	2.8e-08	1.0	4.6e-09	7.2e-10	1.7
st_e07	10	0.0e+00	8.1e-05	0.4	0.0e+00	8.8e-06	3.0
st_jcbpaf2	10	1.1e-07	0.0e+00	2.1	1.1e-07	0.0e+00	2.0
ex2_1_3	13	5.1e-09	3.5e-09	0.5	1.6e-09	1.5e-09	7.7
ex9_1_1	13	0.0	4.5e-06	1.5	0.0	9.2e-07	7.7
ex9_2_3	16	0.0e+00	5.7e-06	2.3	0.0e+00	7.5e-06	49.7
ex2_1_8	24	1.0e-05	0.0e+00	304.6	3.4e-06	0.0e+00	1946.6
ex5_2_2_c1	9	1.0e-2	3.2e+01	1.8	1.6e-05	2.1e-01	2.6
ex5_2_2_c2	9	1.0e-02	7.2e+01	2.1	1.3e-04	2.7e-01	3.5

- ex5_2_2_c1 and ex5_2_2_c2 — Dense is better.
- Sparse attains approx. opt. solutions with the same quality as Dense except ex5_2_2_c1 and ex5_2_2_c2.
- Sparse is much faster than Dense in large dim. cases.

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- **Lasserre's (dense) relaxation**
  — theoretical convergence but expensive in practice.
- **The proposed sparse relaxation**
  = Lasserre's (dense) relaxation + sparsity
  — no theoretical convergence but very powerful in practice.

- **There remain many issues to be studied further.**
  - **Exploiting sparsity.**
  - **Large-scale SDPs.**
  - **Numerical difficulty in solving SDP relaxations of POPs.**

This presentation material is available at

http://www.is.titech.ac.jp/∼kojima/talk.html

Thank you!

References

[1] D. Henrion and J. B. Lasserre, "GloptiPoly: Global optimization over polynomials with Matlab and SeDuMi'.

[2] S. Kim, M. Kojima and H. Waki, "Generalized Lagrangian duals and sums of squares relaxations of sparse polynomial optimization problems". To appear in *SIAM J. on Optimization.*

[3] J. B. Lasserre, "Global optimization with polynomials and the problems of moments", *SIAM J. on Optimization*, 11 (2001) 796–817.

[4] P. A. Parrilo, "Semidefinite programming relaxations for semi algebraic problems". *Math. Prog.*, 96 (2003) 293–320.

[5] S. Prajna, A. Papachristodoulou and P. A. Parrilo, "SOSTOOLS: Sum of Squares Optimization Toolbox for MATLAB – User's Guide".

[6] H. Waki, S. Kim, M. Kojima and M. Muramatsu, "SparsePOP: a Sprase Semidefiite Programming Relaxation of Polynomial Optimization Problems".