Exploiting Sparsity in SOS and SDP Relaxations of Polynomial Optimization Problems

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Outline

1. POPs (Polynomial Optimization Problems)
2. Rough sketch of SOS and SDP relaxations of POPs
3. Exploiting structured sparsity --- unconstrained case
4. Exploiting structured sparsity --- constrained case
5. Numerical results
6. Concluding remarks

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$\mathbb{R}^n$: the n-dim Euclidean space.

$x = (x_1, \ldots, x_n) \in \mathbb{R}^n$: a vector variable.

$f_p(x)$: a multivariate polynomial in $x \in \mathbb{R}^n$ $(p = 0, 1, \ldots, m)$.

POP: min $f_0(x)$ sub.to $f_p(x) \geq 0$ $(p = 1, \ldots, m)$.

Example: $n = 3$

$$\begin{array}{rl} \min & f_0(x) \equiv x_1^3 - 2x_1x_2^2 + x_1^2x_2x_3 - 4x_3^2 \\ \text{sub.to} & f_1(x) \equiv -x_1^2 + 5x_2x_3 + 1 \geq 0, \\ & f_2(x) \equiv x_1^2 - 3x_1x_2x_3 + 2x_3 + 2 \geq 0, \\ & f_3(x) \equiv -x_1^2 - x_2^2 - x_3^2 + 1 \geq 0. \\ & x_1(x_1 - 1) = 0 \text{ (0-1 integer)}, \\ & x_2 \geq 0,\ x_3 \geq 0,\ x_2x_3 = 0 \text{ (complementarity)}. \end{array}$$

- Various problems can be described as POPs.

- A unified theoretical model for global optimization in nonlinear and combinatorial optimization problems.

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[1] J.B.Lasserre, "Global optimization with polynomials and the problems of moments", *SIAM J. on Optim.* (2001).

[2] P.A.Parrilo, "Semidefinite programming relaxations for semialgebraic problems". *Math. Prog.* (2003).

- [1] $\Longrightarrow$ SDP relaxation — primal approach.
- [2] $\Longrightarrow$ SOS relaxation — dual approach.
- [1] and [2] are dual to each other.

(a) Lower bounds for the optimal value.

(b) Convergence to global optimal solutions in theory.

(c) Large-scale SDPs require enormous computation.

(d) SDP[1] + "Exploiting sstructured sparsity"
$\Longrightarrow$ Sparse SDP relaxation

POP: $\min\ f_0(x)$ sub.to $f_p(x) \geq 0\ (p = 1, \ldots, m)$.

Basic idea (practical point of view)

(a) Linearization (Lifting) $\Longrightarrow$ relaxation.

(b) Strengthen the relaxation by adding valid poly. matrix inequalities (before (a)) $\Longrightarrow$ a poly. SDP equiv. to POP.

Represent a polynomial f as $f(x) = \sum_{\alpha \in \mathcal{G}} c(\alpha) x^\alpha$, where

$\mathcal{G}$ = a finite subset of $\mathbb{Z}^n_+ \equiv \{z \in \mathbb{R}^n_+ : z_i \text{ is an integer} \geq 0\}$,
$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ for $\forall x \in \mathbb{R}^n$ and $\forall \alpha \in \mathbb{Z}^n_+$.

Replacing each x^α by a single variable $y_\alpha \in \mathbb{R}$, we have the linearization of $f(x)$: $F(y) = F((y_\alpha : \alpha \in \mathcal{G})) = \sum_{\alpha \in \mathcal{G}} c(\alpha) y_\alpha$.

Example

$$\begin{aligned} f(x_1, x_2) &= 2x_1 - 3x_1^2 + 4x_1 x_2^3 \\ &= 2x^{(1,0)} - 3x^{(2,0)} + 4x^{(1,3)} \\ &\Downarrow \text{(a) Linearization} \\ F(y_{(1,0)}, y_{(2,0)}, y_{(1,3)}) &= 2y_{(1,0)} - 3y_{(2,0)} + 4y_{(1,3)}. \end{aligned}$$

$$\text{POP:} \quad \min\ f_0(x) \quad \text{sub.to} \quad f_p(x) \ge 0\ (p = 1, \ldots, m).$$

Basic idea (practical point of view)

(a) Linearization (Lifting) $\Longrightarrow$ relaxation.

(b) Strengthen the relaxation by adding valid poly. matrix inequalities (before (a)) $\Longrightarrow$ a poly. SDP equiv. to POP.

For $\forall$ finite $\mathcal{G} \subset \mathbb{Z}^n_+ \equiv \{z \in \mathbb{R}^n_+ : z_i$ is an integer $\ge 0\}$, let $u(x;\mathcal{G})$ denote a column vector consisting of x^α $(\alpha \in \mathcal{G})$. Then

(i) rank 1 sym.matrix $u(x;\mathcal{G})u(x;\mathcal{G})^T \succeq O$ for $\forall x \in \mathbb{R}^n$.

(ii) $f_p(x)u(x;\mathcal{G})u(x;\mathcal{G})^T \succeq O$ if $f_p(x) \ge 0$.

Example of (ii). $n = 2$. $\quad \mathcal{G} = \{(0,0),(1,0)\}$.

$$(1-x_1x_2)\begin{pmatrix} 1 \\ x_1 \end{pmatrix}\begin{pmatrix} 1 \\ x_1 \end{pmatrix}^T \succeq O \Leftrightarrow \begin{pmatrix} 1 - x_1x_2 & x_1 - x_1^2x_2 \\ x_1 - x_1^2x_2 & x_1^2 - x_1^3x_2 \end{pmatrix} \succeq O$$

$\Updownarrow$ $\qquad\qquad$ $\Downarrow$ (a) Linearization

$\Downarrow$ (a) Linearization

$$1 - y_{(1,1)} \ge 0 \qquad \begin{pmatrix} 1 - y_{(1,1)} & y_{(1,0)} - y_{(2,1)} \\ y_{(1,0)} - y_{(2,1)} & y_{(2,0)} - y_{(3,1)} \end{pmatrix} \succeq O$$

LMI is stronger!

POP: min $f_0(x)$ sub.to $f_p(x) \geq 0$ $(p = 1, \ldots, m)$.

Basic idea (practical point of view)

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(ii) $f_p(x)u(x;\mathcal{G})u(x;\mathcal{G})^T \succeq O$ if $f_p(x) \geq 0$.

Let $\mathcal{G}_p$ $(p = 1, \ldots, q > m)$ be finite subset of $\mathbb{Z}^n_+$; $0 \in \mathcal{G}_p$.

Polynomial SDP($\mathcal{G}_p$)

$$
\begin{array}{lll}
\min & f_0(x) & \\
\text{sub.to} & f_p(x)u(x,\mathcal{G}_p)u(x,\mathcal{G}_p)^T \succeq O \ (p = 1, \ldots, m) & \Leftarrow \text{(ii)} \\
& u(x,\mathcal{G}_p)u(x,\mathcal{G}_p)^T \succeq O \ (p = m+1, \ldots, q) & \Leftarrow \text{(i)}
\end{array}
$$

Apply (a) $\Rightarrow$ Linear SDP($\mathcal{G}_p$) = SDP relaxation of POP

- $\{\mathcal{G}^k_p\}$; opt.val. of L.SDP($\mathcal{G}^k_p$) $\to$ opt.val. of POP (Lasserre01).
- Expensive $\Rightarrow$ Exploit sparsity of $f_p(x)$ $(p = 0, \ldots, m)$.

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$\mathcal{P}$: $\min_{x \in \mathbb{R}^n} f(x)$, where f is a polynomial with deg $f = 2r$

H : the sparsity pattern of the Hessian matrix of $f(x)$

$$H_{ij} = \begin{cases} \star \text{ if } i = j \text{ or } \partial^2 f(x)/\partial x_i \partial x_j \not\equiv 0, \\ 0 \text{ otherwise.} \end{cases}$$

$f(x)$: correlatively sparse $\Leftrightarrow \exists$ sparse Cholesky fact. of H.

(a) Sparse C.fact. is characterized as a sparse chordal graph $G(N, E')$; $N = \{1, \ldots, n\}$, $E' \supset E = \{(i, j) : H_{ij} = \star\}$.

(b) Let $C_1, C_2, \ldots, C_\ell \subset N$ be the max. cliques of a chordal extension $G(N, E')$ of $G(N, E)$, where $E' = E$ & "fill in".

Sparse relaxation = Linearization of

$$\min\ f(x) \text{ s.t. } u(x, \mathcal{G}_p)u(x, \mathcal{G}_p)^T \succeq O \ (p = 1, 2, \ldots, \ell),$$

where $\mathcal{G}_p \subset \{z \in \mathbb{Z}^n_+ : z_i = 0 \ (i \notin C_p)\}$ $(p = 1, 2, \ldots, \ell)$.

Dense relaxation (Lasserre) = Linearization of

$$\min\ f(x) \text{ s.t. } u(x, \mathcal{G})u(x, \mathcal{G})^T \succeq O, \quad \text{where } \mathcal{G} \subset \mathbb{Z}^n_+.$$

$\mathcal{P}$: $\min_{x \in \mathbb{R}^n} f(x)$, where f is a polynomial with deg $f = 2r$

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$f(x)$: correlatively sparse $\Leftrightarrow$ $\exists$ sparse Cholesky fact. of H.

G. Rosenbrock func: $f(x) = \sum_{i=1}^{n-1} 100(x_{i+1} - x_i^2)^2 + (1 - x_{i+1})^2$.

Dense relaxation (Lasserre) = Linearization of

$$\min \ f(x) \ \text{s.t.} \ u(x, \mathcal{G})u(x, \mathcal{G})^T \succeq O,$$

where $u(x, \mathcal{G}) = (1, x_1, \ldots, x_n, x_1^2, x_1x_2, \ldots, x_2^2, x_2x_3, \ldots, x_n^2)^T$
considting of all monomials in $x_1, \ldots, x_n$ with degree ≤ 2.

- The size of $u(x, \mathcal{G})u(x, \mathcal{G})^T = \binom{n+2}{2}$; $\geq 20,000$ if n=200.
- Difficult to use Dense relaxation for larger POPs in practice.

$\mathcal{P}$: $\min_{x \in \mathbb{R}^n} f(x)$, where f is a polynomial with $\deg f = 2r$

H : the sparsity pattern of the Hessian matrix of $f(x)$

$$H_{ij} = \begin{cases} \star & \text{if } i = j \text{ or } \partial^2 f(x)/\partial x_i \partial x_j \not\equiv 0, \\ 0 & \text{otherwise.} \end{cases}$$

$f(x)$: correlatively sparse $\Leftrightarrow$ $\exists$ sparse Cholesky fact. of H.

G. Rosenbrock func: $f(x) = \sum_{i=1}^{n-1} 100(x_{i+1} - x_i^2)^2 + (1 - x_{i+1})^2$.

- The Hessian matrix is sparse (tridiagonal).

Sparse relaxation = Linearization of

$$\min\ f(x) \text{ s.t. } \begin{pmatrix} 1 \\ x_i \\ x_{i+1} \\ x_i^2 \\ x_i x_{i+1} \\ x_{i+1}^2 \end{pmatrix} \begin{pmatrix} 1 \\ x_i \\ x_{i+1} \\ x_i^2 \\ x_i x_{i+1} \\ x_{i+1}^2 \end{pmatrix}^T \succeq O \ (i = 1, 2, \ldots, n-1),$$

- Much smaller than Dense relaxation; the size is linear in n.

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- This part is complicated, so we present only a basic idea in 3 steps 1), 2) and 3).

POP: min $f_0(x)$ sub.to $f_p(x) \geq 0$ $(p = 1, \ldots, m)$.

Let $\mathcal{G}_p$ $(p = 1, 2, \ldots, q > m)$ be finite subset of $\mathbb{Z}^n_+$; $0 \in \mathcal{G}_p$.

Relaxation = Linearization of Polynomial SDP($\mathcal{G}_p$)

$$\begin{array}{ll} \min & f_0(x) \\ \text{sub.to} & f_p(x) u(x, \mathcal{G}_p) u(x, \mathcal{G}_p)^T \succeq O \ (p = 1, \ldots, m) \quad \text{— (a)} \\ & u(x, \mathcal{G}_p) u(x, \mathcal{G}_p)^T \succeq O \ (p = m+1, \ldots, q) \quad \text{— (b)} \end{array}$$

1) In (a), take $u(x, \mathcal{G}_p)$ so that it shares all x_i's with $f_p(x)$.

For example,

$$-x_1^2 + 2x_5^3 - 2 \geq 0 \Rightarrow (-x_1^2 + 2x_5^3 - 2) \begin{pmatrix} 1 \\ x_1 \\ x_5 \end{pmatrix} \begin{pmatrix} 1 \\ x_1 \\ x_5 \end{pmatrix}^T \succeq O,$$

$$x_3^2 + 3x_3 - 2 \geq 0 \Rightarrow (x_3^2 + 3x_3 - 2) \begin{pmatrix} 1 \\ x_3 \\ x_3^2 \end{pmatrix} \begin{pmatrix} 1 \\ x_3 \\ x_3^2 \end{pmatrix}^T \succeq O.$$

$$\boxed{\text{POP:}\quad \min\ f_0(x)\quad \text{sub.to}\quad f_p(x) \geq 0\ (p = 1, \ldots, m).}$$

Let $\mathcal{G}_p$ $(p = 1, 2, \ldots, q > m)$ be finite subset of $\mathbb{Z}^n_+$; $0 \in \mathcal{G}_p$.

Relaxation = Linearization of Polynomial SDP($\mathcal{G}_p$)

$$\begin{array}{ll} \min & f_0(x) \\ \text{sub.to} & f_p(x) u(x, \mathcal{G}_p) u(x, \mathcal{G}_p)^T \succeq O\ (p = 1, \ldots, m) \quad \text{— (a)} \\ & u(x, \mathcal{G}_p) u(x, \mathcal{G}_p)^T \succeq O\ (p = m+1, \ldots, q) \quad \text{— (b)} \end{array}$$

1) In (a), take $u(x, \mathcal{G}_p)$ so that it shares all x_i's with $f_p(x)$.

2) Let H be the correlative sparsity pattern of $f_0(x)$ and (a);

$$H_{ij} = \begin{cases} \star \text{ if } i = j \text{ or } \partial^2 f_0(x)/\partial x_i \partial x_j \not\equiv 0, \\ \star \text{ if } x_i \text{ and } x_j \text{ involved in } f_p(x) \text{ for some } p, \\ 0 \text{ otherwise.} \end{cases}$$

In (b), choose $u(x, \mathcal{G}_p)$ taking account of the correlative sparsity pattern H as in the unconstrained case.

3) Expand $\mathcal{G}_p$ in (a) as long as the sparsity is maintained.

- Balance degrees of poly. mat. inequalities in (a) and (b).
- Let r denote the max degree of monomials in $u(x, \mathcal{G}_p)$s.
- As r $\uparrow$, a better approx. sol. but the size $\uparrow$.

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- Balance degrees of poly. mat. inequalities in (a) and (b).
- Let r denote the max degree of monomials in $u(x, \mathcal{G}_p)$s.
- As r ↑, a better approx. sol. but the size ↑.

r = relaxation order

Software

- **MATLAB for constructing sparse and dense SDP relaxation problems**
- **SeDuMi to solve SDPs.**

Hardware

- **2.4GHz Xeon cpu with 6.0GB memory.**

G.Rosenbrock function:

$$f(x) = \sum_{i=2}^{n} \left(100(x_i - x_{i-1}^2)^2 + (1 - x_i)^2\right)$$

- Two minimizers on $\mathbb{R}^n$: $x_1 = \pm 1,\ x_i = 1\ (i \geq 2)$.
- Sparse can not handle multiple minimizers effectively.
- Perturb the function or add $x_1 \geq 0 \Rightarrow$ unique minimizer.

cpu in sec.				cpu in sec.	
Sparse	ϵ_{obj}	n	ϵ_{obj}	Sparse	Dense
0.2	5.1e-04	10	2.5e-08	0.2	10.6
0.3	1.8e-03	15	6.5e-08	0.2	756.6
2.2	3.1e-03	200	5.2e-07	2.2	—
4.6	5.9e-03	400	2.5e-06	3.7	—
8.6	8.3e-03	800	5.5e-06	6.8	—

$$\epsilon_{\text{obj}} = \frac{|\text{the lower bound for opt. value} - \text{the approx. opt. value}|}{\max\{1, |\text{the lower bound for opt. value}|\}}.$$

An optimal control problem from Coleman et al. 1995

$$\left.\begin{array}{l} \min \ \dfrac{1}{M}\displaystyle\sum_{i=1}^{M-1}(y_i^2+x_i^2) \\ \text{s.t.} \quad y_{i+1}=y_i+\dfrac{1}{M}(y_i^2-x_i), \quad (i=1,\ldots,M-1), \quad y_1=1. \end{array}\right\}$$

Numerical results on sparse relaxation ($r=2$)

M	# of variables	ϵ_{obj}	ϵ_{feas}	cpu
600	1198	3.4e-08	2.2e-10	3.4
700	1398	2.5e-08	8.1e-10	3.3
800	1598	5.9e-08	1.6e-10	3.8
900	1798	1.4e-07	6.8e-10	4.5
1000	1998	6.3e-08	2.7e-10	5.0

$$\epsilon_{\text{obj}}=\frac{|\text{the lower bound for opt. value} - \text{the approx. opt. value}|}{\max\{1, |\text{the lower bound for opt. value}|\}},$$

ϵ_{feas} = the maximum error in the equality constraints,

cpu : cpu time in sec. to solve an SDP relaxation problem.

alkyl.gms : a benchmark problem from globallib

$$\begin{array}{ll} \min & -6.3x_5x_8 + 5.04x_2 + 0.35x_3 + x_4 + 3.36x_6 \\ \text{sub.to} & -0.820x_2 + x_5 - 0.820x_6 = 0, \\ & 0.98x_4 - x_7(0.01x_5x_{10} + x_4) = 0, \\ & -x_2x_9 + 10x_3 + x_6 = 0, \\ & x_5x_{12} - x_2(1.12 + 0.132x_9 - 0.0067x_9^2) = 0, \\ & x_8x_{13} - 0.01x_9(1.098 - 0.038x_9) - 0.325x_7 = 0.574, \\ & x_{10}x_{14} + 22.2x_{11} = 35.82, \\ & x_1x_{11} - 3x_8 = -1.33, \\ & \text{lbd}_i \le x_i \le \text{ubd}_i \ (i = 1, 2, \ldots, 14). \end{array}$$

			Sparse			Dense (Lasserre)		
problem	n	r	ϵ_{obj}	ϵ_{feas}	cpu	ϵ_{obj}	ϵ_{feas}	cpu
alkyl	14	2	4.1e-03	2.7e-01	0.9	6.3e-06	1.8e-02	17.6
alkyl	14	3	5.6e-10	2.0e-08	6.9	—	—	—

$r =$ relaxation order,

$$\epsilon_{\text{obj}} = \frac{|\text{the lower bound for opt. value} - \text{the approx. opt. value}|}{\max\{1, |\text{the lower bound for opt. value}|\}},$$

$\epsilon_{\text{feas}} =$ the maximum error in the equality constraints,

cpu : cpu time in sec. to solve an SDP relaxation problem.

Some other benchmark problems from globallib

			Sparse			Dense (Lasserre)		
problem	n	r	ϵ_{obj}	ϵ_{feas}	cpu	ϵ_{obj}	ϵ_{feas}	cpu
ex3_1_1	8	3	6.3e-09	4.7e-04	5.5	0.7e-08	2.5e-03	597.8
st_bpaf1b	10	2	3.8e-08	2.8e-08	1.0	4.6e-09	7.2e-10	1.7
st_e07	10	2	0.0e+00	8.1e-05	0.4	0.0e+00	8.8e-06	3.0
ex2_1_3	13	2	5.1e-09	3.5e-09	0.5	1.6e-09	1.5e-09	7.7
ex9_1_1	13	2	0.0	4.5e-06	1.5	0.0	9.2e-07	7.7
ex9_2_3	16	2	0.0e+00	5.7e-06	2.3	0.0e+00	7.5e-06	49.7
ex2_1_8	24	2	1.0e-05	0.0e+00	304.6	3.4e-06	0.0e+00	1946.6
ex5_2_2_c1	9	2	1.0e-2	3.2e+01	1.8	1.6e-05	2.1e-01	2.6
ex5_2_2_c1	9	3	6.4e-04	2.3e-01	295.9	-	-	-
ex5_2_2_c2	9	2	1.0e-02	7.2e+01	2.1	1.3e-04	2.7e-01	3.5
ex5_2_2_c2	9	3	5.8e-04	8.9e-01	332.9	-	-	-

$r =$ relaxation order,

$$\epsilon_{\mathrm{obj}} = \frac{|\text{the lower bound for opt. value} - \text{the approx. opt. value}|}{\max\{1, |\text{the lower bound for opt. value}|\}},$$

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Some other benchmark problems from globallib

			Sparse			Dense (Lasserre)		
problem	n	r	ϵ_{obj}	ϵ_{feas}	cpu	ϵ_{obj}	ϵ_{feas}	cpu
ex3_1_1	8	3	6.3e-09	4.7e-04	5.5	0.7e-08	2.5e-03	597.8
st_bpaf1b	10	2	3.8e-08	2.8e-08	1.0	4.6e-09	7.2e-10	1.7
st_e07	10	2	0.0e+00	8.1e-05	0.4	0.0e+00	8.8e-06	3.0
ex2_1_3	13	2	5.1e-09	3.5e-09	0.5	1.6e-09	1.5e-09	7.7
ex9_1_1	13	2	0.0	4.5e-06	1.5	0.0	9.2e-07	7.7
ex9_2_3	16	2	0.0e+00	5.7e-06	2.3	0.0e+00	7.5e-06	49.7
ex2_1_8	24	2	1.0e-05	0.0e+00	304.6	3.4e-06	0.0e+00	1946.6
ex5_2_2_c1	9	2	1.0e-2	3.2e+01	1.8	1.6e-05	2.1e-01	2.6
ex5_2_2_c1	9	3	6.4e-04	2.3e-01	295.9	-	-	-
ex5_2_2_c2	9	2	1.0e-02	7.2e+01	2.1	1.3e-04	2.7e-01	3.5
ex5_2_2_c2	9	3	5.8e-04	8.9e-01	332.9	-	-	-

- ex5_2_2_c1 and ex5_2_2_c2 ($r = 2$) — Dense is better.
- Sparse attains approx. opt. solutions with the same quality as Dense except ex5_2_2_c1 and ex5_2_2_c2.
- Sparse is much faster than Dense in large dim. and higher relaxation order cases.

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- Lasserre's (dense) relaxation
  — theoretical convergence but expensive in practice.
- The proposed sparse relaxation
  = Lasserre's (dense) relaxation + sparsity
  — no theoretical convergence but very powerful in practice.

- There remain many issues to be studied further.
  - Exploiting sparsity.
  - Large-scale SDPs.
  - Numerical difficulty in solving SDP relaxations of POPs.

This presentation material is available at

http://www.is.titech.ac.jp/∼kojima/talk.html

Thank you!