Parallel implementation of polyhedral continuation methods for systems of polynomial equations

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Contents

1. A system of polynomial equations $f(x) = 0$, where

$$\begin{aligned} x &= (x_1, x_2, \ldots, x_n) \in \mathbb{C}^n, \\ f(x) &= (f_1(x), f_2(x), \ldots, f_n(x)), \\ f_j(x) &= \text{a polynomial in } n \text{ complex variables } x_1, x_2, \ldots, x_n. \end{aligned}$$

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Example

$$\begin{aligned} & n = 3,\ x = (x_1, x_2, x_3),\ f(x) = (f_1(x), f_2(x), f_3(x)), \\ & f_1(x_1, x_2, x_3) = x_1^2 - (2.1 + i)x_1 x_2 x_3^2 + 8.5, \\ & f_2(x_1, x_2, x_3) = 1.5x_1^2 x_2 - x_1^2 x_2^2 x_3 - 1.6, \\ & f_3(x_1, x_2, x_3) = (3.6 + i)x_1 x_2^3 + 4.3x_1 x_2^2 x_3^2. \end{aligned}$$

Find all isolated solutions in $\mathbb{C}^n$.

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Find all isolated solutions in $\mathbb{C}^n$.

- A Fundamental problem in numerical mathematics.
- Various engineering applications.
- Global optimization.

2. Typical benchmark test problem — 1: Economic-n polynomial:

$$
\begin{aligned}
&(x_1 + x_1x_2 + x_2x_3 + \cdots + x_{n-2}x_{n-1})x_n - 1 = 0 \\
&(x_2 + x_1x_3 + \cdots + x_{n-3}x_{n-1})x_n - 2 = 0 \\
&\cdots \\
&(x_{n-2} + x_1x_{n-1})x_n - (n-2) = 0 \\
&x_{n-1}x_n - (n-1) = 0 \\
&x_1 + x_2 + \cdots + x_{n-1} + 1 = 0.
\end{aligned}
$$

n	♯ of isolated solutions
10	256
11	512
12	1,024
13	2,048
	$\cdots$
20	262,144
	$\cdots$
n	2^{n-2}

Typical benchmark test problem — 2: Cyclic-n polynomial

$$
\begin{aligned}
f_1(x) &= x_1 + x_2 + \cdots + x_n, \\
f_2(x) &= x_1 x_2 + x_2 x_3 + \cdots + x_n x_1, \\
&\dots \\
f_{n-1}(x) &= x_1 x_2 \dots x_{n-1} + x_2 x_3 \dots x_n + \cdots + x_n x_1 \cdots x_{n-1}, \\
f_n(x) &= x_1 x_2 \dots x_{n-1} x_n - 1.
\end{aligned}
$$

(i) Symmetric structur — invariance under the cyclic permutation.

(ii) ♯ of sol? & ↑↑. (iii) ∃ singular sol and sol comp with dim > 0.

n	♯ of nonsingular isolated solutions	♯/n	♯/($2n$)
10	34,940	3,494	1,747
11	184,756	16,796	8,398
12	367,488	30,624	15,312
13	2,696,044	207,288	103,694
	...		

(i) ⇒ We can reduce the solutions to be computed to $1/n$ (or $1/(2n)$).

Enormous computational power for solving large scale problems

⇒ Parallel computation

3. Rough sketch of the polyhedral homotopy method

• Based on Bernshtein's theory on bounding the number of solutions of a polynomail system in terms of its mixed volume. [Bernshtein '75]

• Currently the most powerful and practical method for computing all solutions of a system of polynomial equations.

PHCpack [Verschelde '96], [Li '99], [Dai-Kim-Kojima '01], etc.

• Suitable for parallel computation;
all solutions can be computed independently in parallel.

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3. Rough sketch of the polyhedral homotopy method — 2

Phase 1. Construct a family of homotopy functions.

- Branch-and-bound methods.
- Large scale linear programs.

Phase 2. Trace homotopy paths by predictor-corrector methods.

- Highly nonlinear homotopy paths that require complicated techniques for step length control.

Phase 3. Verify that all isolated solutions are computed.

- The number of solutions is unknown in general.
- Approximate solutions are computed but exact solutions are never computed.

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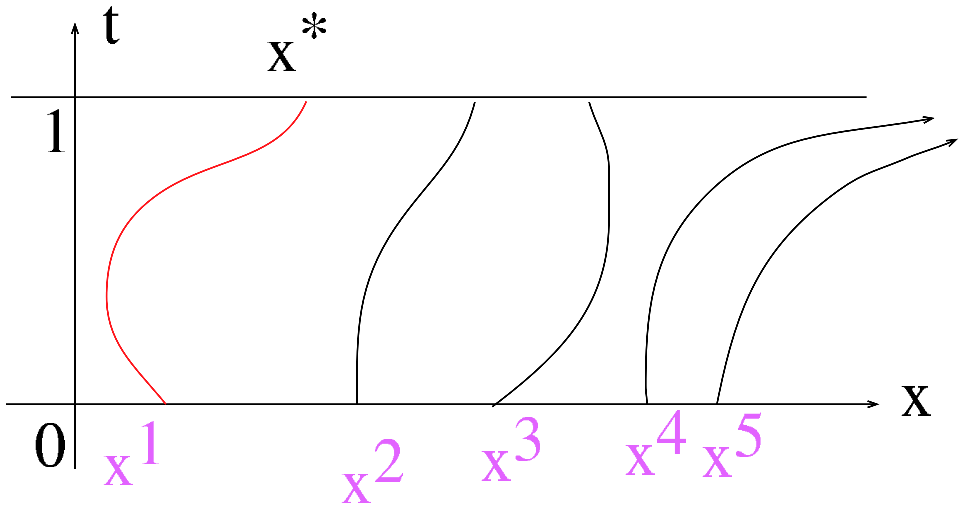
4. Basic ideas of Phases 1 and Phase 2.

Phase 1. Construct a homotopy system $h(x,t) = 0$ such that

(i) all solutions of the initial sys $h(x,0) = 0$ are known,

(ii) $h(x,1) = f(x)$ for $\forall x \in \mathbb{C}^n$; if $h(x,1) = 0$, x is a sol of $f(x) = 0$,

(iii) each solution x^* of $f(x) = 0$ is connected to a solution x^1 of $h(x,0) = 0$ through a solution path of $h(x,t) = 0$.



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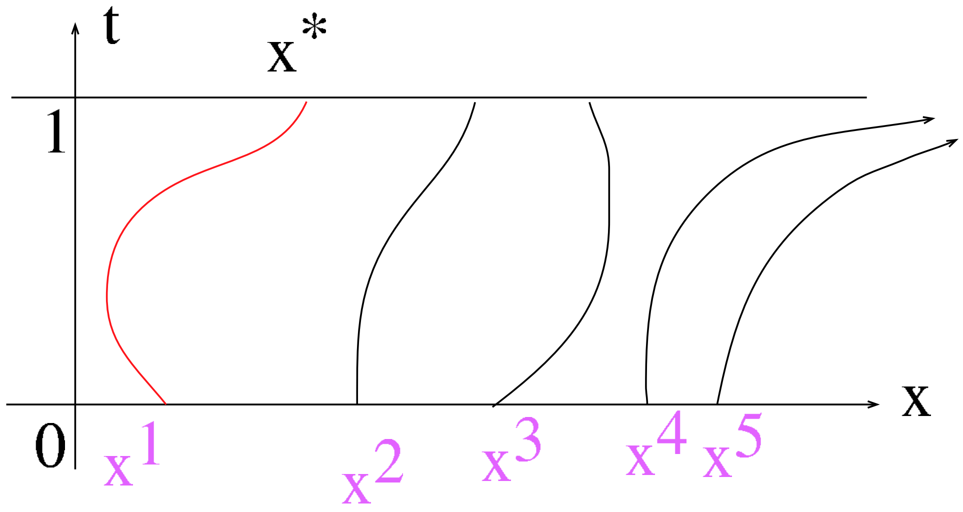

Phase 2. Starting from each known sol of the initial sys $h(x,0) = 0$, we trace the solution paths of $h(x,t) = 0$ till t reaches 1 by a predictor-corrector method to obtain a solution of $f(x) = 0$.

- **This idea is common for the traditional linear homotopy method and the polyhedral homotopy method.**

- **Some solution paths diverge as $t \longrightarrow 1$; tracing such paths are useless.**

- **The number of useless divergent paths is much less in the polyhedral homotopy method than in the traditional homotopy method.**

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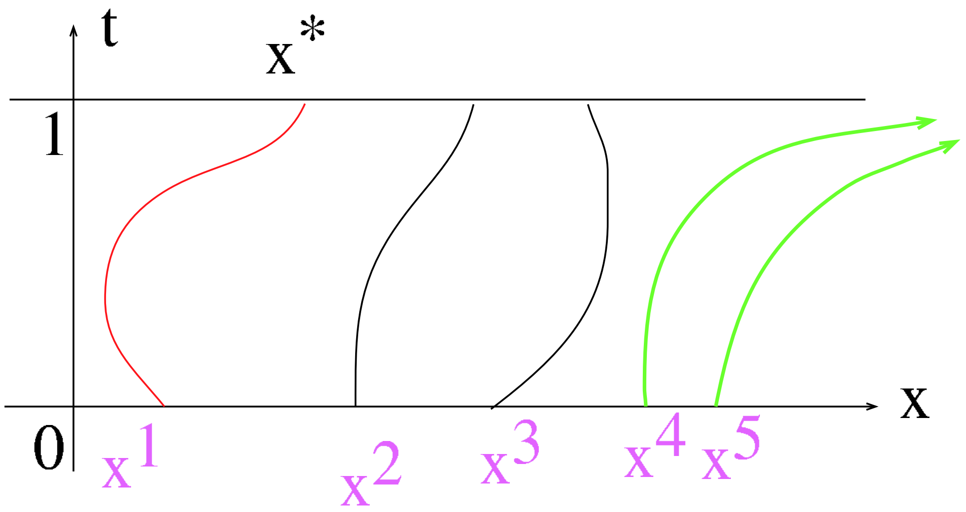

- **Multiple homotopy functions are employed in polyhedral homotopy methods while a common single homotopy function is employed for all solutions of $f(x) = 0$ in the traditional linear homotopy method.**

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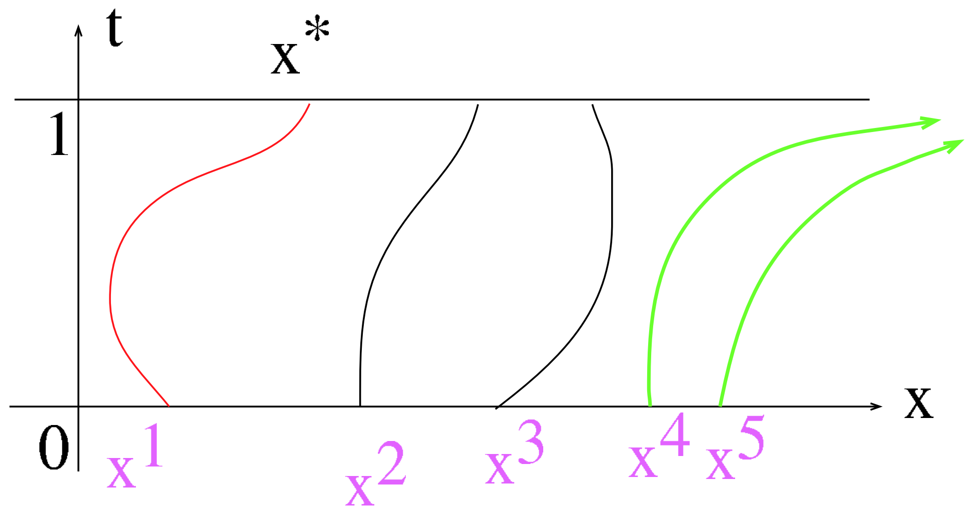

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Notation

For $\forall \boldsymbol{a} \in \mathbb{Z}_+^n \equiv \{(a_1, \ldots, a_n) \geq 0 : a_j \text{ is integer}\}$, $\forall \boldsymbol{x} \in \mathbb{C}^n$, let

$$\boldsymbol{x}^{\boldsymbol{a}} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}.$$

Write $\forall\ f_j(\boldsymbol{x})$ of a poly. system $\boldsymbol{f}(\boldsymbol{x}) = (f_1(\boldsymbol{x}), \ldots, f_n(\boldsymbol{x}))$ as

$$f_j(\boldsymbol{x}) = \sum_{\boldsymbol{a} \in \mathcal{A}_j} c_j(\boldsymbol{a}) \boldsymbol{x}^{\boldsymbol{a}},$$

where $c_j(\boldsymbol{a}) \in \mathbb{C}$ $(\boldsymbol{a} \in \mathcal{A}_j)$ and $\mathcal{A}_j$ a finite subset of $\mathbb{Z}_+^n$ $(j = 1, \ldots, n)$. We call $\mathcal{A}_j$ the support of $f_j(\boldsymbol{x})$.

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For example, $n = 3$,

$$\begin{aligned} f_3(x_1, x_2, x_3) &= (3.6 + i)x_1 x_2^3 + 4.3 x_1 x_2^2 x_3^2 \\ &= c_3((1,3,0))\boldsymbol{x}^{(1,3,0)} + c_3((1,2,2))\boldsymbol{x}^{(1,2,2)} \\ &= \sum_{\boldsymbol{a} \in \mathcal{A}_3} c_3(\boldsymbol{a}) \boldsymbol{x}^{\boldsymbol{a}} \end{aligned}$$

where $\mathcal{A}_3 = \{(1,3,0),\ (1,2,2)\}$,

$c_3((1,3,0)) = 3.6 + i,\ c_3((1,2,2)) = 4.3.$

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The main part (construction of a family of polyhedral homotopy functions) of Phase 1 is reduced to the following combinatorial problem.

Choose $\omega_j(a) \in \mathbb{R}$ (randomly) $(a \in \mathcal{A}_j,\ j = 1,2,\ldots,n)$.

Find all $(\alpha,\beta) \in \mathbb{R}^{2n}$ satisfying

(1) $\langle a,\alpha\rangle + \omega_j(a) - \beta_j \geq 0$ $(a \in \mathcal{A}_j,\ j = 1,\ldots,n)$,

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Illustration of (1) and (2): $n = 4$, **a variable vector** $(\alpha, \beta) \in \mathbb{R}^8$

$$
(1) \quad \begin{cases} \langle a, \alpha \rangle + \omega_1(a) - \beta_1 \geq 0 \;\; (a \in \mathcal{A}_1), \\ \langle a, \alpha \rangle + \omega_2(a) - \beta_2 \geq 0 \;\; (a \in \mathcal{A}_2), \\ \langle a, \alpha \rangle + \omega_3(a) - \beta_3 \geq 0 \;\; (a \in \mathcal{A}_3), \\ \langle a, \alpha \rangle + \omega_4(a) - \beta_4 \geq 0 \;\; (a \in \mathcal{A}_4). \end{cases}
$$

(2) requires that exactly two equalities hold in each group $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$.

Choose $\omega_j(a) \in \mathbb{R}$ **(randomly)** $(a \in \mathcal{A}_j,\; j = 1, 2, \ldots, n)$.

Find all $(\alpha, \beta) \in \mathbb{R}^{2n}$ **satisfying**

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- **Parallel computation.**
- **The simplex method for linear programs.**
- **Implicit enum. tech. (or b-and-b. methods) used in optimization.**

$$\Uparrow$$

This problem forms an important subprob. in Phase 1.

$$\Uparrow$$

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Polyhedral homotopy system

$$(3)\ h_j(x,t) \equiv \sum_{a\in\mathcal{A}_j} c_j(a) x^{a} t^{\rho_j(a)} = 0,\ (x,t) \in \mathbb{C}^n \times [0,1]\ (j = 1,\dots,n)$$

$$h(x,1) \equiv f(x),\ h(x,0) = 0 : \text{ a binomial system } \Downarrow$$

$\Uparrow$ Phase 2 - Tracing homotopy paths by pred.-correct. method

$\Uparrow$

$\Uparrow$ Each solution (α, β) induces a homotopy function.

$$\rho_j(a;\alpha,\beta,\omega) \equiv \langle a,\alpha\rangle + \omega_j(a) - \beta_j \geq 0\ (a \in \mathcal{A}_j,\ j = 1,\dots,n)$$

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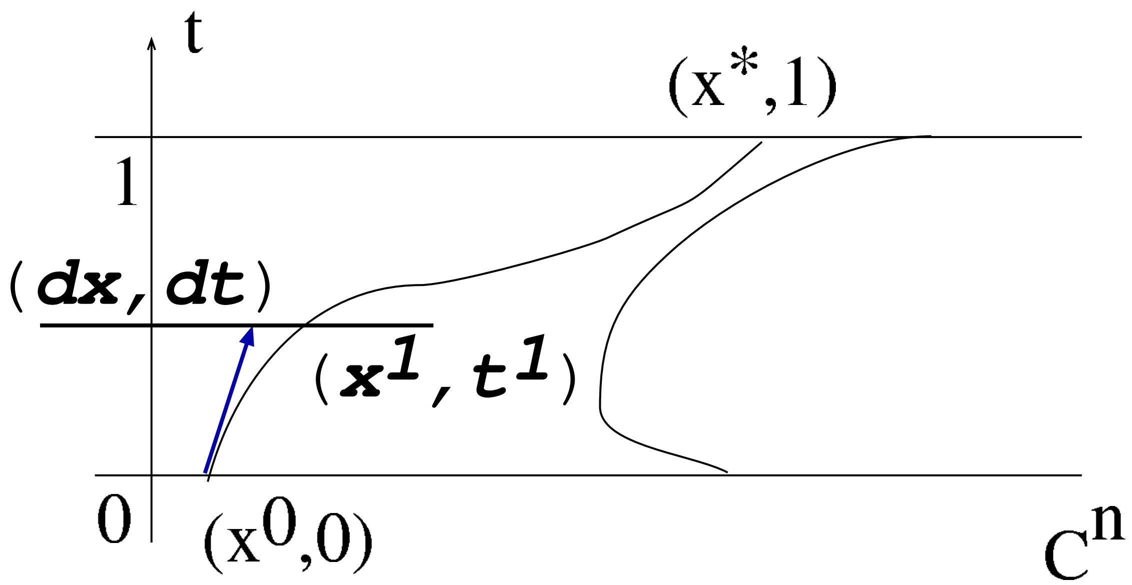

Pred. with a step len. $dt > 0$: $Dh_x(x^0, 0)dx + Dh_t(x^0, 0)dt = 0$

Corr. Newton meth. to $h(x, 0 + dt) = 0$ from $\tilde{x}^0 = x^0 + dx$.

Polyhedral homotopy system

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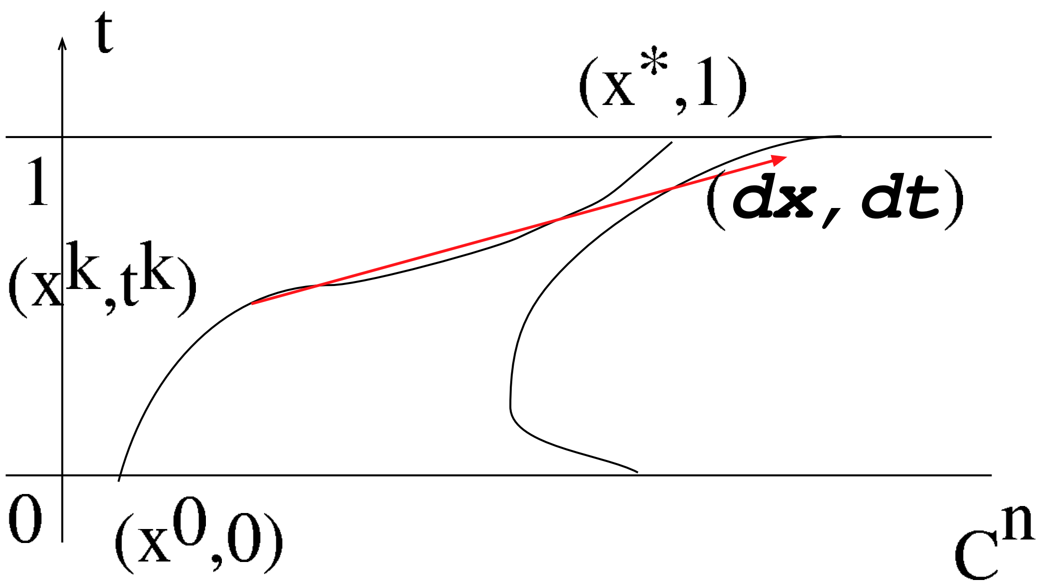

Predictor with $dt > 0$ at (x^k, t^k): $Dh_x(x^k, t^k)dx + Dh_t(x^k, t^k)dt = 0$

Too large step length $dt \Longrightarrow$ Jump into a different solution path.

Too small step length $dt \Longrightarrow$ more pred. iter. and more cpu time.

Step length control is essential!

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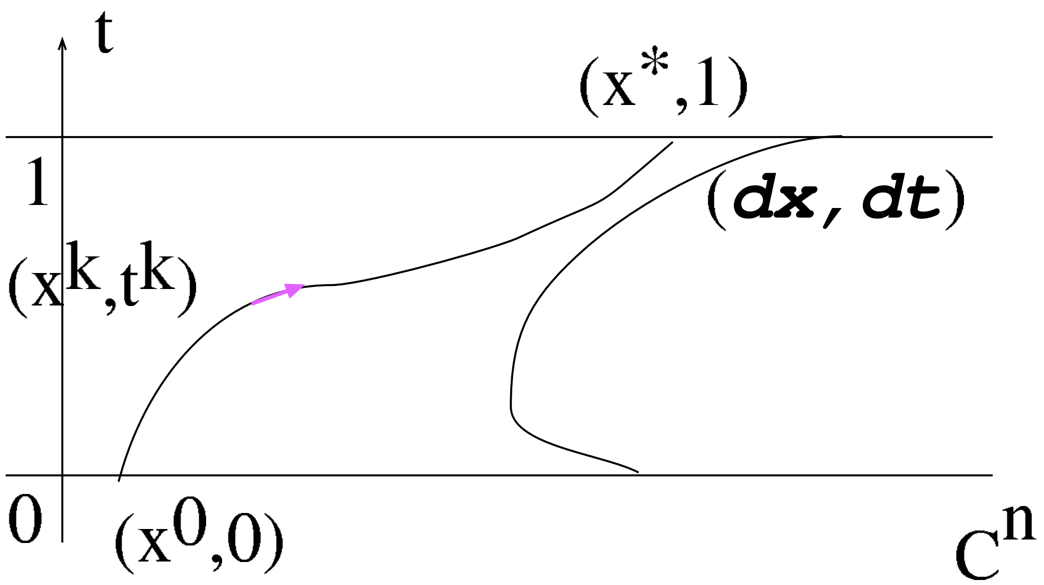

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Difficulty in Phase 2 — High nonlinearity in $h(x,t)$. Some $\rho_j(a)$'s are huge, for example

$$h_j(x,t)\ =\ \cdots + c_j(a)x^{a}t^{10} + c_j(a')x^{a'}t^{1,000} + c_j(a")x^{a"}t^{100,000} + \cdots$$

- **Complicated step length control.**
- **Construct homotopies with less power $\Longrightarrow$ Opt. problem.**

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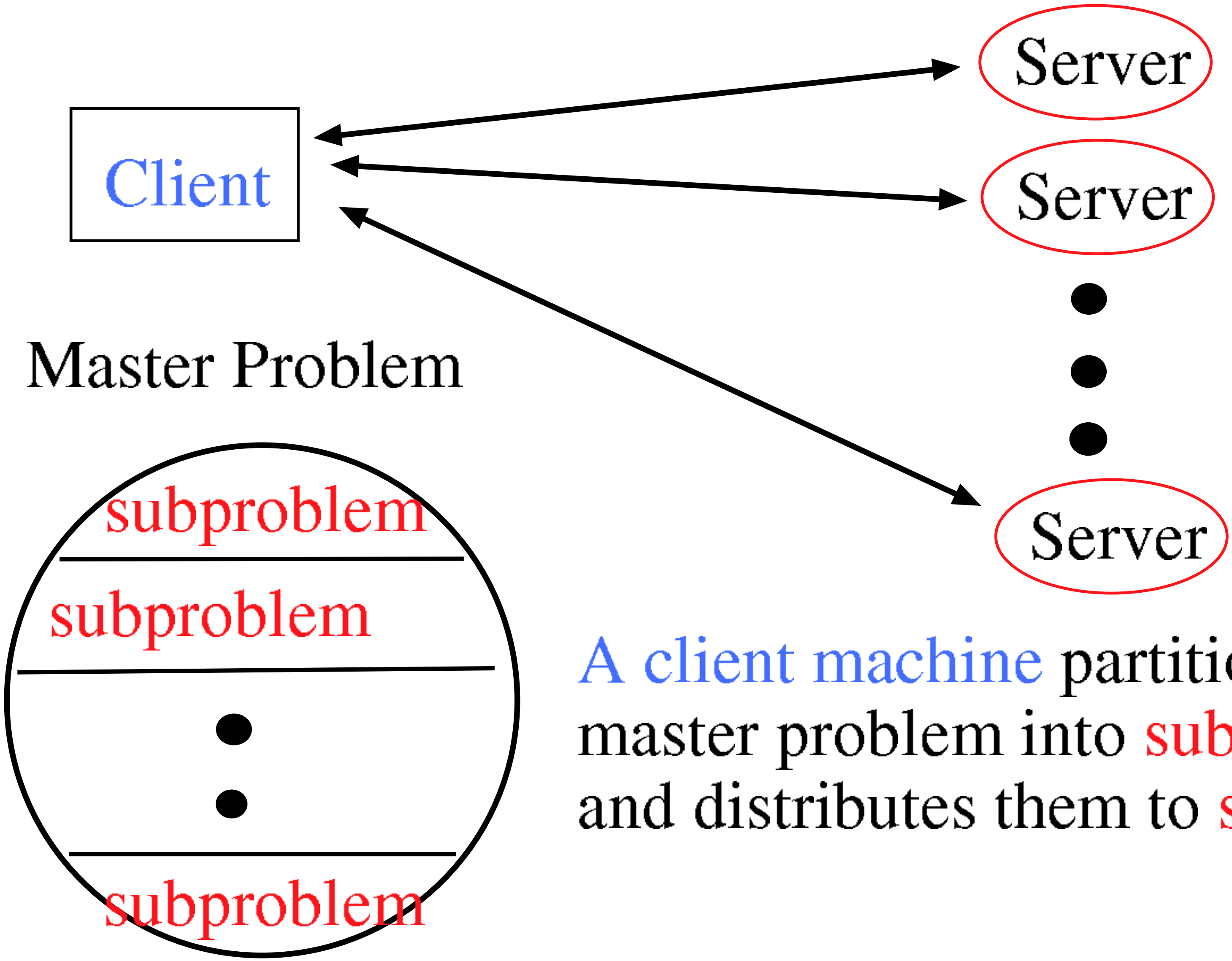
Change of t^p as $t \to 1$, $p = 10,\ 1,000,\ 10,000$

t	t^{10}	$t^{1,000}$	$t^{100,000}$
1.0 - 1.0e-01	1.0 - 6.51e-01	0.0	0.0
1.0 - 1.0e-02	1.0 - 9.56e-02	0.0	0.0
1.0 - 1.0e-03	1.0 - 9.96e-03	1.0 - 6.32e-01	0.0
1.0 - 1.0e-04	1.0 - 1.00e-03	1.0 - 9.52e-02	0.0
1.0 - 1.0e-05	1.0 - 1.00e-04	1.0 - 9.95e-03	1.0 - 6.32e-01
1.0 - 1.0e-06	1.0 - 1.00e-05	1.0 - 1.00e-03	1.0 - 9.52e-02
1.0 - 1.0e-07	1.0 - 1.00e-06	1.0 - 1.00e-04	1.0 - 9.95e-03

7. Numerical results on parallel implementation of Phases 1 and 2

- **Ninf: Client-Server Computing System by Sekiguchi, et. al.**



A client machine partitions a master problem into subproblems and distributes them to server machines

Find all $(\alpha, \beta) \in \mathbb{R}^{2n}$ satisfying

(1) $\langle a, \alpha \rangle + \omega_j(a) - \beta_j \geq 0$ $(a \in \mathcal{A}_j,\ j = 1, \ldots, n)$,

(2) for $\forall j$, exactly 2 of $\{\langle a, \alpha \rangle + \omega_j(a) - \beta_j : a \in \mathcal{A}_j\}$ are 0.

Parallel Comp. of all solutions of (1) & (2) — Eco-n problems

Intel Pentium III 824MHz

# CPUs	Eco-n Problems, real time in second				
	$n = 12$	$n = 13$ (speed-up-ratio)		$n = 14$ (speed-up-ratio)	
1	1,379	8,399	(1.00)		
2	686	4,200	(2.00)		
4	344	2,106	(3.99)		
8	181	1,064	(7.89)	12,500	(1.00)
16	97	553	(15.19)	6,471	(1.93)
32	66	287	(29.06)	3,339	(3.74)
64		177	(47.11)	1,779	(7.03)
# solutions of (1) & (2)	364	719		1,227	

Find all $(\alpha, \beta) \in \mathbb{R}^{2n}$ satisfying

(1) $\langle a, \alpha \rangle + \omega_j(a) - \beta_j \geq 0$ $(a \in \mathcal{A}_j,\ j = 1, \ldots, n)$,

(2) for $\forall j$, exactly 2 of $\{\langle a, \alpha \rangle + \omega_j(a) - \beta_j : a \in \mathcal{A}_j\}$ are 0.

Parallel Comp. of all solutions of (1) & (2) — Cyc-n problems

Intel Pentium III (824MHz)

# CPUs	Cyclic-n Problems, real time in second				
	$n = 11$	$n = 12$	(speed-up-ratio)	$n = 13$	(speed-up-ratio)
1	1,647	14,403	(1.00)		
2	832	7,214	(2.00)		
4	414	3,624	(3.97)		
8	214	1,825	(7.89)		
16	107	926	(15.55)	11,745	(1.00)
32	67	476	(30.26)	5,888	(1.99)
64		276	(52.18)	3,155	(3.72)
128		182	(79.14)	1,841	(6.38)
# solutions of (1) & (2)	13,101	29,561		144,517	

The entire Phase 1

(a) All solutions of (1) & (2)

(b) Reduction of the powers of the parameter $t \in [0, 1]$
$\Rightarrow$ A large scale Linear Program;
$\#$ variables ≤ 200 and $1,000,000 \geq \#$ inequalities
$\Rightarrow$ Cutting plane methods based on the dual simplex method

(c) All solutions of initial binomial systems
$\Rightarrow$ starting solutions for Phase 2

# CPUs	Cyclic-12, real time in second				
	(a)	(b)	(c)	(a)+(b)+(c)	speed-up-ratio
1	28,033	546	380	28,959	1.00
2	14,125	306	191	14,622	1.98
4	7,342	187	104	7,633	3.79
8	3,793	123	56	3,972	7.29
16	2,166	88	52	2,316	12.50
32	1,390	68	43	1,521	19.04

Celeron 500MHz

Phase 2 — Tracing solution paths

Celeron 500MHz

Athlon 1600MHz

# CPUs	Cyclic-n, real time in second					
	$n = 11$	(sp-up-ratio)	$n = 12$	(sp-up-ratio)	$n = 13$	(sp-up-ratio)
2	47,345	(1.00)				
4	23,674	(2.00)				
8	11,852	(3.99)				
16	5,927	(7.99)				
32	2,967	(15.96)				
64	1,487	(31.84)	2,592	(1.00)		
128			1,332	(1.95)	10,151	(1.00)
256			703	(3.69)	5,191	(1.95)
# paths traced	16,796		41,696		208,012	

cyclic11, $n = 11$
# of paths traced = 16196; all paths converged
# of nonsingular solutions = 16196*n = 184756
# of isolated singular solutions = 0

cyclic12, $n = 12$
# of paths traced = 41696; some paths diverged?
# of nonsingular solutions = 30624*n = 367488
# of isolated singular solutions with multiplicity 5 = 48*n = 576
# of isolated singular solutions with multiplicity 10 = 4*n = 48
Some nonisolated solutions — solution components with dim $\geq$ 1 ?

cyclic13, $n = 13$
# of paths traced = 208012; all paths converged
# of nonsingular solutions = 207388*n = 2696044
# of isolated singular solutions with multiplicity 4 = 156*n = 2028

8. Concluding Remarks — 1

(a) While we trace a homotopy path numerically, a jump into another path sometime occurs $\Longrightarrow$ Not 100% reliable. But the reliability is very high; for example, less than 0.1% solutions are missing in our numerical experiments. There are some ways to overcome such flaw.

8. Concluding Remarks — 1

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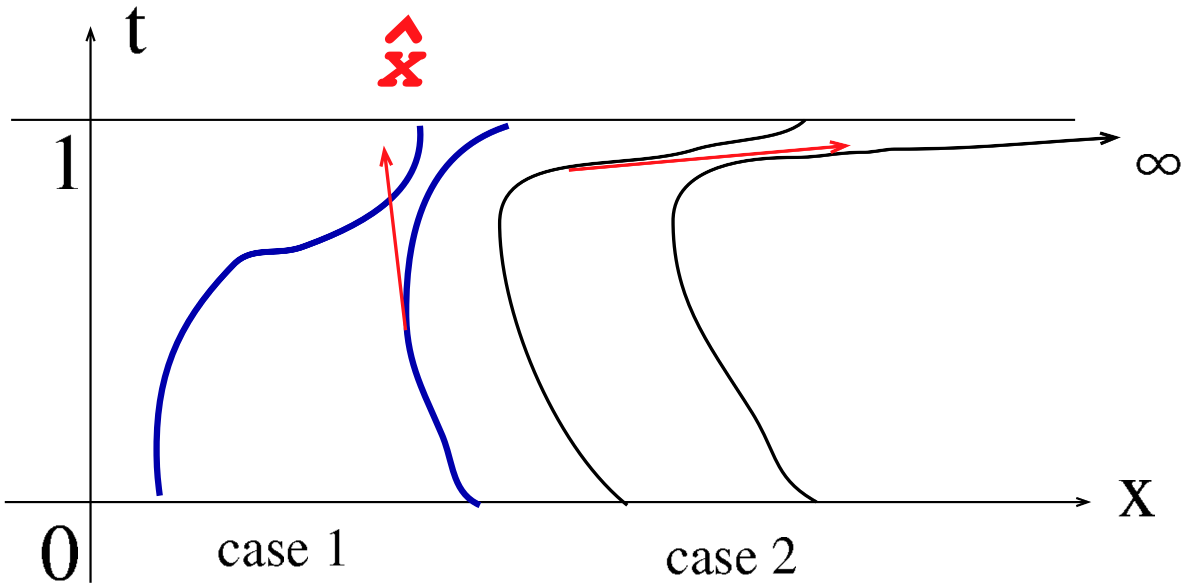
Suppose that numerical tracing of two paths led to a common solution $\hat{x}$ as in case 1 below $\Rightarrow$ an illegal jump while tracing one of them. In such cases, follow again those two paths using smaller predictor step lengths.



8. Concluding Remarks — 2

(b) **Reducing the powers of the continuation parameter t is crucial to achieve the numerical stability and efficiency in tracing homotopy paths. This problem can be formulated as a nonlinear combinatorial optimization problem.**

(c) **The polyhedral homotopy continuation method involves various optimization techniques such as branch-and-bound methods, linear programs, and predictor-corrector methods.**

(d) **An important feature of the homotopy continuation method is that all homotopy paths can be computed independently and simultaneously in parallel.**

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