A General Framework for Convex Relaxation of Polynomial Optimization Problems over Cones

Masakazu Kojima, Sunyoung Kim and Hayato Waki

IMA Workshop: Semidefinite Programming and Robust Optimization.

March 15, 2003

Outline

1. **Convex relaxation of global optimization problems**

2. **An illustrative example**

3. **Polynomial optimization problems over cones and their linearization**

4. **General framework for convex relaxation**

5. **Basic theory**

6. **Concuding remarks**

1. Convex relaxation of global optimization problems — 2

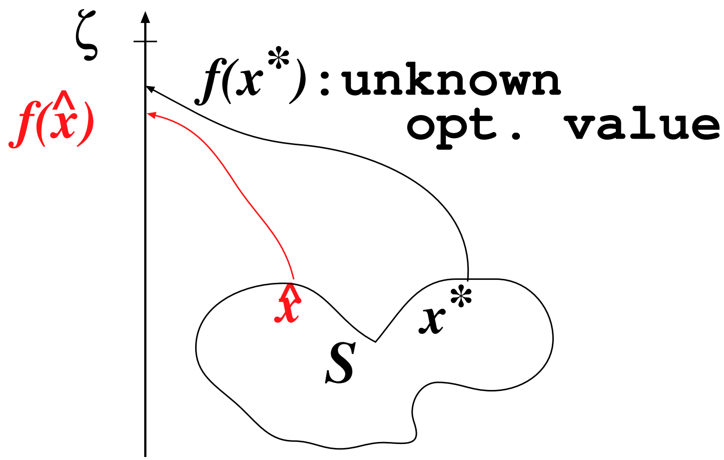
(1) max. $f(x)$ sub.to $x \in S$, where $f : \mathbb{R}^n \to \mathbb{R}$ and $S \subset \mathbb{R}^n$.

(a) a feasible solution $\hat{x} \in S$ with a larger objective value $f(\hat{x})$

(b) a smaller upper bound ζ for the unknown optimal value $f(x^*)$

$\Longrightarrow$ a main role of convex relaxation

If $\zeta - f(\hat{x})$ is smaller, we can accept $\hat{x}$ as a higher quality approximate optimal solution.



1. Convex relaxation of global optimization problems — 3

(1) max. $f(x)$ sub.to $x \in S$, where $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and $S \subset \mathbb{R}^n$.

(a) a feasible solution $\hat{x} \in S$ with a larger objective value $f(\hat{x})$

(b) a smaller upper bound ζ for the unknown optimal value $f(x^*)$

$\Longrightarrow$ a main role of convex relaxation

- SDP relaxation is very powerful in theory.
  - (a) Lovász-Schrijver'91 for 0-1 IPs
  - (b) Goemans-Willianson'95 for max-cut problems
  - (c) Some special QOPs can be solved approximately or exactly by SDP relaxation, Nesterov'88, Ye'99, Zhang'00, Ye-Zhang'01
  - (d) Successive convex relaxation of nonconvex set, Kojima-Tuncel'00 — Extension of (a) to QOPs.
  - (e) Hierarchical SDP relaxation by Lasserre'01, Parrilo for polynomial programs — theoretically powerful: optimal values and solutions can be computed by solving a finite number of SDP relaxations.
  - (f) . . .

(1) max. $f(x)$ sub.to $x \in S$, where $f : \mathbb{R}^n \to \mathbb{R}$ and $S \subset \mathbb{R}^n$.

(a) a feasible solution $\hat{x} \in S$ with a larger objective value $f(\hat{x})$

(b) a smaller upper bound ζ for the unknown optimal value $f(x^*)$

$\Longrightarrow$ a main role of convex relaxation

- Can SDP (or convex) relaxation, without combining any technique on (a), be powerful enough to solve practical large scale problems?

???, mainly because solving large scale SDPs accurately is expensive .

- Incorporate convex relaxation into traditional opt. methods.
- How to combine them effectively.
- Exploration of effective and inexpensive convex relaxations.

Besides SDP and LP relaxation, we explore various convex relaxations towards practically effective and efficient methods.

The purpose of this talk is to present

a general and flexible framework for convex relaxation methods

The main ingredients are:

(a) Polynomial Optimization Problems $\supset$ QOPs and 0-1 IPs

$\Downarrow$(b) Add valid constraints and reformulate

(c) Polynomial Optimization Problems over Cones

$\Downarrow$ (d) Linearization (Lifting)

(e) Linear Optimization Problems over Cones

I will talk about

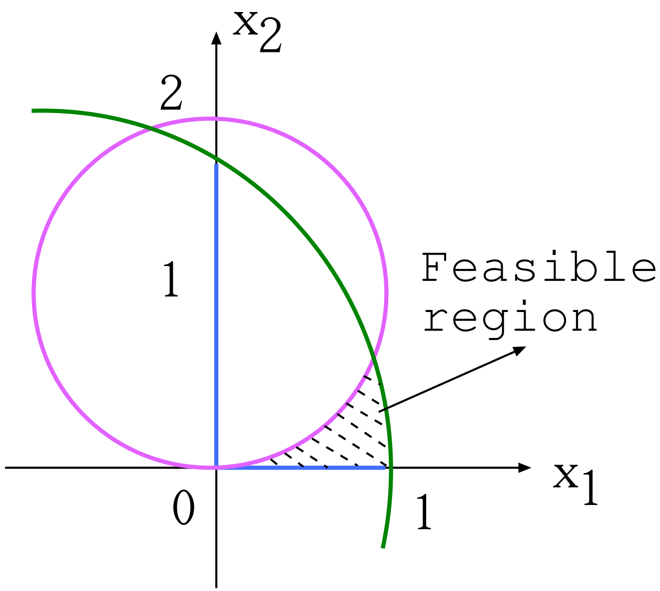
- **An illustrative example**
- **(c) $\Rightarrow$ (d) $\Rightarrow$ (e)**
- **(b)**

2. An illustrative example — 1

Original problem: max. $-2x_1 + x_2$

sub.to $x_1 \geq 0,\ x_2 \geq 0,\ x_1^2 + x_2^2 - 2x_2 \geq 0,$

$$\left\| \begin{pmatrix} x_1 + 1 \\ x_2 \end{pmatrix} \right\| \leq 2 \text{ (SOCP constraint)}$$



2. An illustrative example — 4

Original problem:

$$\begin{array}{ll} \text{max.} & -2x_1 + x_2 \\ \text{sub.to} & x_1 \geq 0,\ x_2 \geq 0,\ x_1^2 + x_2^2 - 2x_2 \geq 0, \\ & \left\| \begin{pmatrix} x_1 + 1 \\ x_2 \end{pmatrix} \right\| \leq 2 \ \text{(SOCP constraint)} \end{array}$$

$\Downarrow$ **Valid constraints and/or reformulation**

$$\begin{array}{ll} \text{max.} & -2x_1 + x_2 \\ \text{sub.to} & x_1 \geq 0,\ x_2 \geq 0,\ x_1^2 \geq 0,\ x_1 x_2 \geq 0,\ x_2^2 \geq 0,\ x_1^2 + x_2^2 - 2x_2 \geq 0, \\ & \left\| \begin{pmatrix} x_1 + 1 \\ x_2 \end{pmatrix} \right\| \leq 2,\ \left\| \begin{pmatrix} x_1^2 + x_1 \\ x_1 x_2 \end{pmatrix} \right\| \leq 2x_1,\ \left\| \begin{pmatrix} x_1 x_2 + x_2 \\ x_2^2 \end{pmatrix} \right\| \leq 2x_2. \end{array}$$

$\Downarrow$ **Linearization: Keep the linear terms,**
but replace each nonlinear term by a single independent variable

$$\begin{array}{ll} \text{max.} & -2x_1 + x_2 \\ \text{sub.to} & x_1 \geq 0,\ x_2 \geq 0,\ X_{11} \geq 0,\ X_{12} \geq 0,\ X_{22} \geq 0, \\ & X_{11} + X_{22} - 2x_2 \geq 0, \\ & \left\| \begin{pmatrix} x_1 + 1 \\ x_2 \end{pmatrix} \right\| \leq 2,\ \left\| \begin{pmatrix} X_{11} + x_1 \\ X_{12} \end{pmatrix} \right\| \leq 2x_1,\ \left\| \begin{pmatrix} X_{12} + x_2 \\ X_{22} \end{pmatrix} \right\| \leq 2x_2. \end{array}$$

2. An illustrative example — 5

Original problem: max. $-2x_1 + x_2$

sub.to $x_1 \geq 0,\ x_2 \geq 0,\ x_1^2 + x_2^2 - 2x_2 \geq 0,$

$\left\| \begin{pmatrix} x_1 + 1 \\ x_2 \end{pmatrix} \right\| \leq 2$ (SOCP constraint)

$\Downarrow$ Valid constraints and/or reformulation

$$\begin{array}{ll} \text{max.} & -2x_1 + x_2 \\ \text{sub.to} & x_1 \geq 0,\ x_2 \geq 0,\ x_1^2 \geq 0,\ x_1 x_2 \geq 0,\ x_2^2 \geq 0,\ x_1^2 + x_2^2 - 2x_2 \geq 0, \\ & \left\| \begin{pmatrix} x_1 + 1 \\ x_2 \end{pmatrix} \right\| \leq 2,\ \left\| \begin{pmatrix} x_1^2 + x_1 \\ x_1 x_2 \end{pmatrix} \right\| \leq 2x_1,\ \left\| \begin{pmatrix} x_1 x_2 + x_2 \\ x_2^2 \end{pmatrix} \right\| \leq 2x_2. \end{array}$$

$\Uparrow \quad X_{11} = x_1 x_1, X_{12} = x_1 x_2, X_{22} = x_2 x_2$

$$\begin{array}{ll} \text{max.} & -2x_1 + x_2 \\ \text{sub.to} & x_1 \geq 0,\ x_2 \geq 0,\ X_{11} \geq 0,\ X_{12} \geq 0, X_{22} \geq 0, \\ & X_{11} + X_{22} - 2x_2 \geq 0, \\ & \left\| \begin{pmatrix} x_1 + 1 \\ x_2 \end{pmatrix} \right\| \leq 2, \left\| \begin{pmatrix} X_{11} + x_1 \\ X_{12} \end{pmatrix} \right\| \leq 2x_1, \left\| \begin{pmatrix} X_{12} + x_2 \\ X_{22} \end{pmatrix} \right\| \leq 2x_2. \end{array}$$

3. Polynomial opt. problems over cones and their linearization — 3

$$\max.\ f_0(x) \text{ sub.to } f(x) \in \mathcal{K}, \quad \text{where}$$

$\mathcal{K}$: a closed convex cone in $\mathbb{R}^m$,
$x = (x_1, \ldots, x_n)$: a variable vector, $f(x) \equiv (f_1(x), \ldots, f_m(x))$,
$f_j(x)$: a polynomial in $x_1, \ldots, x_n$ $(j = 0, 1, \ldots, m)$.

Typical examples of $\mathcal{K}$: $\mathbb{R}^m_+$: the nonnegative orthant of $\mathbb{R}^m$.

$\mathbb{S}^\ell_+$: the cone of $\ell \times \ell$ psd symmetric matrices, where we identify each $\ell \times \ell$ matrix as an $\ell \times \ell$ dim vector.

$$\mathbb{N}^{1+\ell}_p \equiv \left\{ v = (v_0, v_1, \ldots, v_\ell) \in \mathbb{R}^{1+\ell} : \left(\sum_{i=1}^{\ell} |v_i|^p \right)^{1/p} \leq v_0 \right\}$$

: the pth order cone $(p \geq 1)$.

$\mathbb{N}^{1+\ell}_2$: the second order cone.

When $f_j(x)$ $(j = 0, 1, 2, \ldots, m)$ are linear,

$$\mathcal{K} = \mathbb{S}^\ell_+ \Rightarrow \text{SDP (Semidefinite Program)},$$

$$\mathcal{K} = \mathbb{N}^{1+\ell}_2 \Rightarrow \text{SOCP (Second-Order Cone Program)}$$

$$\max.\ f_0(x) \text{ sub.to } f(x) \in \mathcal{K}, \quad \text{where}$$

$\mathcal{K}$: a closed convex cone in $\mathbb{R}^m$,

$x = (x_1, \ldots, x_n)$: a variable vector, $f(x) \equiv (f_1(x), \ldots, f_m(x))$,

$f_j(x)$: a polynomial in $x_1, \ldots, x_n$ $(j = 0, 1, \ldots, m)$.

Linearization —	Keep the linear terms, but replace each nonlinear term by a single independent variable.

Example 1: $n = 2,\ m = 2.$

$$f(x_1, x_2) = \begin{pmatrix} 1 - 2x_1 + 3x_2 + 4x_1^2 + 5x_1x_2 + 6x_2^2 \\ 9 + 8x_1 + 7x_2 + 6x_1^2 - 5x_1x_2 - 4x_2^2 \end{pmatrix} \in \mathcal{K}$$

$\Downarrow$ Linearization

$$F(x_1, x_2, X_{11}, X_{12}, X_{22}) = \begin{pmatrix} 1 - 2x_1 + 3x_2 + 4X_{11} + 5X_{12} + 6X_{22} \\ 9 + 8x_1 + 7x_2 + 6X_{11} - 5X_{12} - 4X_{22} \end{pmatrix} \in \mathcal{K}$$

Here the three new variables X_{11}, X_{12} and X_{22} are introduced.

$$\text{max. } f_0(x) \text{ sub.to } f(x) \in \mathcal{K}, \quad \text{where}$$

$\mathcal{K}$: a closed convex cone in $\mathbb{R}^m$,

$x = (x_1, \ldots, x_n)$: a variable vector, $f(x) \equiv (f_1(x), \ldots, f_m(x))$,

$f_j(x)$: a polynomial in $x_1, \ldots, x_n$ $(j = 0, 1, \ldots, m)$.

Linearization —	**Keep the linear terms, but replace each nonlinear term by a single independent variable.**

Example 2: $n = 3$, $m = 2$.

$$f(x_1, x_2, x_3) = \begin{pmatrix} 1 - 2x_1 + 3x_2 + 4x_1^2 x_3 + 5x_1 x_2 x_3 + 6x_3^4 \\ 9 + 8x_1 + 7x_2 + 6x_1^2 x_3 - 5x_1 x_2 x_3 - 4x_3^4 \end{pmatrix} \in \mathcal{K}$$

$\Downarrow$ Linearization

$$F(x_1, x_2, U, V, W) = \begin{pmatrix} 1 - 2x_1 + 3x_2 + 4U + 5V + 6W \\ 9 + 8x_1 + 7x_2 + 6U - 5V - 4W \end{pmatrix} \in \mathcal{K}$$

Here the new variables U, V and W are introduced. In general, we need a systematic method of assigning a new variable to each nonlinear term.

$$\max.\ f_0(x) \text{ sub.to } f(x) \in \mathcal{K}, \quad \text{where}$$

$\mathcal{K}$: a closed convex cone in $\mathbb{R}^m$,
$x = (x_1, \ldots, x_n)$: a variable vector, $f(x) \equiv (f_1(x), \ldots, f_m(x))$,
$f_j(x)$: a polynomial in $x_1, \ldots, x_n$ $(j = 0, 1, \ldots, m)$.

Linearization —	**Keep the linear terms, but replace each nonlinear term by a single independent variable.**

Systematic method of assigning a new variable to each nonlinear term:

$$\text{a nonlinear term } x_1^{\alpha} x_2^{\beta} \cdots x_n^{\zeta} \Rightarrow y_{(\alpha,\beta,\ldots,\zeta)} \in \mathbb{R} \text{ a new variable}$$

(Sherali et.al, Lasserre'01, ...). For example,

$$n = 5,\ x_1^2 x_2 x_3^3 x_5^4 = x_1^2 x_2^1 x_3^3 x_4^0 x_5^4 \Rightarrow y_{(2,1,3,0,4)}.$$

In theory, any method of assigning a new variable to each nonlinear term works. $\Rightarrow$ This method is not essential.

4. General framework for convex relaxation — 3

Original QOP, 0-1 IP, Polynomial programs to be solved

$\Downarrow$ **Valid constraints and/or reformulate**

POP: max. $f_0(x)$ sub.to $f(x) \in \mathcal{K}$, where

$\mathcal{K}$: a closed convex cone in $\mathbb{R}^m$,
$x = (x_1, \ldots, x_n)$: a variable vector, $f(x) \equiv (f_1(x), \ldots, f_m(x))$,
$f_j(x)$: a polynomial in $x_1, \ldots, x_n$ $(j = 0, 1, \ldots, m)$.

$\Downarrow$ **Linearization — Keep the linear terms, but replace each**
$\Downarrow$ **nonlinear term by a single independent variable.**

LOP: max. $F_0(x, y)$ sub.to $F(x, y) \in \mathcal{K}$, where

y denotes a new variable vector whose elements correspond to nonlinear terms appeared in the polynomials $f_j(x)$ $(j = 0, 1, \ldots, m)$.

Illustrative example again — 2

Original problem: max. $-2x_1 + x_2$

sub.to $x_1 \geq 0,\ x_2 \geq 0,\ x_1^2 + x_2^2 - 2x_2 \geq 0,$

$\left\| \begin{pmatrix} x_1 + 1 \\ x_2 \end{pmatrix} \right\| \leq 2$ (SOCP constraint)

⇓ Valid constraints and/or reformulation

$$
\begin{array}{ll}
\text{max.} & -2x_1 + x_2 \\
\text{sub.to} & x_1 \geq 0,\ x_2 \geq 0,\ x_1^2 \geq 0,\ x_1 x_2 \geq 0,\ x_2^2 \geq 0,\ x_1^2 + x_2^2 - 2x_2 \geq 0, \\
& \left\| \begin{pmatrix} x_1 + 1 \\ x_2 \end{pmatrix} \right\| \leq 2,\ \left\| \begin{pmatrix} x_1^2 + x_1 \\ x_1 x_2 \end{pmatrix} \right\| \leq 2x_1,\ \left\| \begin{pmatrix} x_1 x_2 + x_2 \\ x_2^2 \end{pmatrix} \right\| \leq 2x_2.
\end{array}
$$

⇓ Linearization: Keep the linear terms, but replace each nonlinear term by a single independent variable

$$
\begin{array}{ll}
\text{max.} & -2x_1 + x_2 \\
\text{sub.to} & x_1 \geq 0,\ x_2 \geq 0,\ y_{20} \geq 0,\ y_{11} \geq 0, y_{02} \geq 0, \\
& y_{20} + y_{02} - 2x_2 \geq 0, \\
& \left\| \begin{pmatrix} x_1 + 1 \\ x_2 \end{pmatrix} \right\| \leq 2, \left\| \begin{pmatrix} y_{20} + x_1 \\ y_{11} \end{pmatrix} \right\| \leq 2x_1, \left\| \begin{pmatrix} y_{11} + x_2 \\ y_{02} \end{pmatrix} \right\| \leq 2x_2.
\end{array}
$$

Illustrative example again — 4

Original problem: max. $-2x_1 + x_2$

sub.to $x_1 \geq 0,\ x_2 \geq 0,\ x_1^2 + x_2^2 - 2x_2 \geq 0,$

$$\left\| \begin{pmatrix} x_1 + 1 \\ x_2 \end{pmatrix} \right\| \leq 2 \text{ (SOCP constraint)}$$

⇓ Valid constraints and/or reformulation

$$\begin{array}{ll} \text{max.} & -2x_1 + x_2 \\ \text{sub.to} & x_1 \geq 0,\ x_2 \geq 0,\ x_1^2 + x_2^2 - 2x_2 \geq 0, \\ & \left\| \begin{pmatrix} x_1 + 1 \\ x_2 \end{pmatrix} \right\| \leq 2,\ \begin{pmatrix} 1 \\ x_1 \\ x_2 \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 \end{pmatrix} \equiv \begin{pmatrix} 1 & x_1 & x_2 \\ x_1 & x_1^2 & x_1 x_2 \\ x_2 & x_1 x_2 & x_2^2 \end{pmatrix} \succeq O. \end{array}$$

⇓ Linearization

$$\begin{array}{ll} \text{max.} & -2x_1 + x_2 \qquad \text{— SDP} \\ \text{sub.to} & x_1 \geq 0,\ x_2 \geq 0,\ y_{20} + y_{02} - 2x_2 \geq 0, \\ & \left\| \begin{pmatrix} x_1 + 1 \\ x_2 \end{pmatrix} \right\| \leq 2,\ \begin{pmatrix} 1 & x_1 & x_2 \\ x_1 & y_{20} & y_{11} \\ x_2 & y_{11} & y_{02} \end{pmatrix} \succeq O. \end{array}$$

Given a problem, there are various ways of adding valid constraints and reformulating the problem. They usually yield different convex relaxations.

Illustrative example again — 5

Original problem:
$$\begin{array}{ll} \text{max.} & -2x_1 + x_2 \\ \text{sub.to} & x_1 \geq 0,\ x_2 \geq 0,\ x_1^2 + x_2^2 - 2x_2 \geq 0, \\ & \left\| \begin{pmatrix} x_1 + 1 \\ x_2 \end{pmatrix} \right\| \leq 2 \text{ (SOCP constraint)} \end{array},$$

we obtained two distinct convex relaxations.

$$\begin{array}{ll} \text{max.} & -2x_1 + x_2 \qquad \text{— SOCP} \\ \text{sub.to} & x_1 \geq 0,\ x_2 \geq 0,\ y_{20} \geq 0,\ y_{11} \geq 0,\ y_{02} \geq 0, \\ & y_{20} + y_{02} - 2x_2 \geq 0, \\ & \left\| \begin{pmatrix} x_1 + 1 \\ x_2 \end{pmatrix} \right\| \leq 2, \left\| \begin{pmatrix} y_{20} + x_1 \\ y_{11} \end{pmatrix} \right\| \leq 2x_1, \left\| \begin{pmatrix} y_{11} + x_2 \\ y_{02} \end{pmatrix} \right\| \leq 2x_2. \end{array}$$

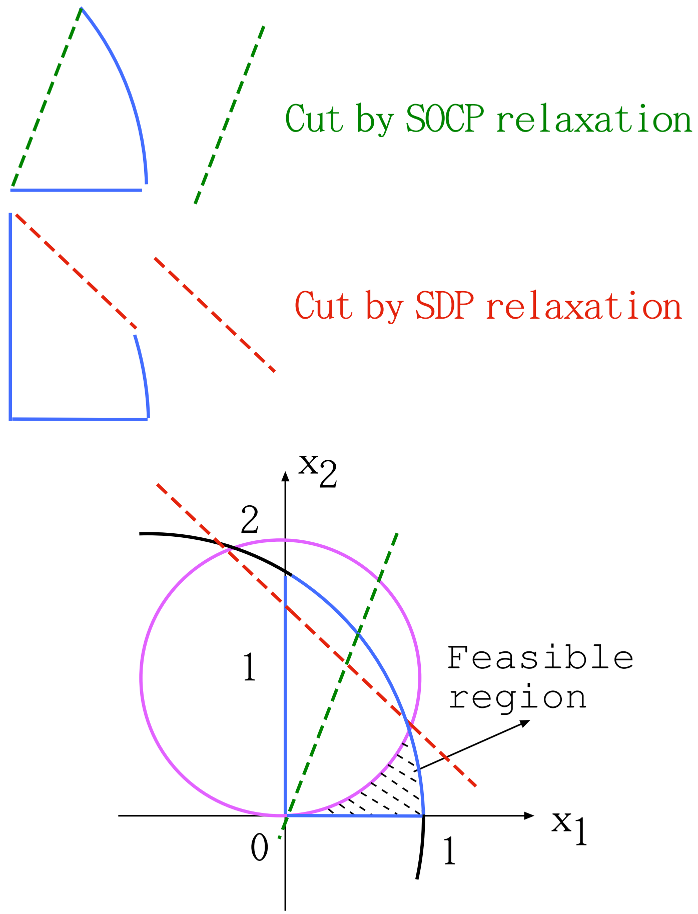
$$\begin{array}{ll} \text{max.} & -2x_1 + x_2 \qquad \text{— SDP} \\ \text{sub.to} & x_1 \geq 0,\ x_2 \geq 0,\ y_{20} + y_{02} - 2x_2 \geq 0, \\ & \left\| \begin{pmatrix} x_1 + 1 \\ x_2 \end{pmatrix} \right\| \leq 2, \begin{pmatrix} 1 & x_1 & x_2 \\ x_1 & y_{20} & y_{11} \\ x_2 & y_{11} & y_{02} \end{pmatrix} \succeq O. \end{array}$$

Illustrative example again — 6

Original problem: max. $-2x_1 + x_2$

sub.to $x_1 \geq 0,\ x_2 \geq 0,\ x_1^2 + x_2^2 - 2x_2 \geq 0,$

$\left\| \begin{pmatrix} x_1 + 1 \\ x_2 \end{pmatrix} \right\| \leq 2$ (SOCP constraint) ,



Some examples of valid constraints — 2

- **Universally valid constraints.**

(a) SDP type:

$$u(x)^T u(x) = \begin{pmatrix} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \\ x_1 & x_1^2 & x_1x_2 & x_1^3 & x_1^2x_2 & x_1x_2^2 \\ x_2 & x_1x_2 & x_2^2 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\ x_1^2 & x_1^3 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\ x_1x_2 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\ x_2^2 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & x_2^4 \end{pmatrix} \succeq O,$$

where $u(x) = \begin{pmatrix} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \end{pmatrix}$

More generally, take a row vector consisting of a basis of the polynomials in $x_1, \ldots, x_n$ with degree ℓ for $u(x)$. [Lasserre'01].

(b) SOCP (Second-Order Cone Programming) type:

$$\forall\ f_1, f_2 : \mathbb{R}^n \to \mathbb{R},\ \left\| \begin{pmatrix} f_1(x)^2 - f_2(x)^2 \\ 2f_1(x)f_2(x) \end{pmatrix} \right\| \leq f_1(x)^2 + f_2(x)^2$$

Some examples of valid constraints — 4

- **Deriving valid constraints, "multiplication" of valid constraints:**

$$\begin{array}{rcl} \text{original constraints} & & \text{new constraints} \\ \mathbb{R} \ni f(x) \geq 0,\ \mathbb{R} \ni g(x) \geq 0 & \Rightarrow & f(x)g(x) \geq 0 \text{ [Sherali et.al'92]} \\ f(x) \geq 0,\ G(x) \succeq O & \Rightarrow & f(x)G(x) \succeq 0 \text{ [Lasserre'01]} \end{array}$$

$$\begin{array}{rcl} F(x) \succeq O,\ G(x) \succeq O & \Rightarrow & F(x) \otimes G(x) \succeq 0 \text{ (Kronecker product)} \\ \left.\begin{array}{l} \|f(x)\| \leq f_0(x),\ f(x) \in \mathbb{R}^\ell \\ \|g(x)\| \leq g_0(x),\ g(x) \in \mathbb{R}^\ell \end{array}\right\} & \Rightarrow & \|f(x) \circ g(x)\| \leq f_0(x) g_0(x) \\ \text{(SOCP constraints)} & & \text{(component-wise product)} \end{array}$$

5. Basic theory — 3

POP: max. $f_0(x)$ sub.to $f(x) \in \mathcal{K}$, where
$\mathcal{K}$: a closed convex cone in $\mathbb{R}^m$, $f(x) \equiv (f_1(x), \ldots, f_m(x))$.

$\Downarrow$ Linearization

LOP: max. $F_0(x, y)$ sub.to $F(x, y) \in \mathcal{K}$, where y denotes a new variable vector corresponding to nonlinear terms of $f_j(x)$ $(j = 0, \ldots, m)$.

Lagrangian funct: $L(x, v) \equiv f_0(x) + \langle v, f(x) \rangle$ for $\forall x \in \mathbb{R}^n, v \in \mathcal{K}^*$

Under the Slater condition ($\exists x$; $f(x) \in$ int $\mathcal{K}$), if $\bar{\zeta}$ is the opt. value of LOP then there exists $\bar{v} \in \mathcal{K}^*$ satisfying $L(x, \bar{v}) = \bar{\zeta}$ for $\forall x \in \mathbb{R}^n$.

$$\text{Hence } \bar{\zeta} = \max\{L(x, \bar{v}) : x \in \mathbb{R}^n\} \text{ (a Lagrangian relaxation)}$$
$$\geq \min_{v \in \mathcal{K}^*} \max\{L(x, v) : x \in \mathbb{R}^n\} \text{ (Lagrangian dual relaxation)}$$

- Lagrangian dual relaxation is stronger
- Given $v \in \mathcal{K}^*$, $L(x, v)$ is not concave in general.
- In the standard SDP relaxation to QOP, LOP $\approx$ Lagrangian dual.

POP: max. $c^T x$ sub.to $f(x) \in \mathcal{K}$, where

$\mathcal{K}$: a closed convex cone in $\mathbb{R}^m$, $f(x) \equiv (f_1(x), \ldots, f_m(x))$.

$\Downarrow$ Linearization

LOP: max. $c^T x$ sub.to $F(x, y) \in \mathcal{K}$, where y denotes a new variable vector corresponding to nonlinear terms of $f_j(x)$ $(j = 0, \ldots, m)$.

$\Updownarrow$

LOP': max. $c^T x$ sub.to $x \in \widehat{\mathcal{F}} \equiv \{x \in \mathbb{R}^n : F(x, y) \in \mathcal{K}$ for some $y\}$, where $\widehat{\mathcal{F}}$ denotes the projected feasible region of LOP onto $\mathbb{R}^n$:

Define $\mathcal{L} \equiv \{v \in \mathcal{K}^* : \langle v, f(x) \rangle$ is linear in $x \in \mathbb{R}^n\}$ and

$$\widetilde{\mathcal{F}} \equiv \{x \in \mathbb{R}^n : \langle v, f(x) \rangle \geq 0 \text{ for every } v \in \mathcal{L}\}$$

"the set of linear consequences of $f(x) \in \mathcal{K}$" .

Then $\widehat{\mathcal{F}} \subseteq \widetilde{\mathcal{F}}$, and (the closure of $\widehat{\mathcal{F}}$) $= \widetilde{\mathcal{F}}$ under $\exists x$; $f(x) \in$ int $\mathcal{K}$.

6. Concluding remarks

The framework proposed in this talk for convex relaxation is quite general.

But we need to investigate various issues to deal with large scale problems.

- **Effectiveness — How do we generate better bounds?**
- **Low cost — Resulting relaxed problems need to be solved cheaply.**
- **How to combine this framework with other methods like the branch-and-bound method.**
- **Exploiting structure; sparsity, separability, (partial) linearlity, (partial) convexity — Intuitively, we only have to take account of nonconvex variables (or directions).**
- **Parallel computation.**