多項式最適化問題の半正定値計画緩和
— 疎性の活用 —

最適化：モデリングとアルゴリズム
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Outline

1. POPs (Polynomial Optimization Problems)
2. How do we formulate structured sparsity?
3. Sparse SDP relaxation of POPs
4. Numerical results
5. Concluding remarks.

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Notation and Symbols

$\mathbb{R}^n$: the n-dim Euclidean space.
$\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$: a vector variable.
$f_j(\boldsymbol{x})$: a multivariate polynomial in $\boldsymbol{x} \in \mathbb{R}^n$ $(j = 0, 1, \ldots, m)$.

> POP: min $f_0(\boldsymbol{x})$ sub.to $f_j(\boldsymbol{x}) \geq 0$ $(j = 1, \ldots, m)$.

Example: $n = 3$

$$
\begin{array}{rl}
\text{min} & f_0(\boldsymbol{x}) \equiv x_1^3 - 2x_1x_2^2 + x_1^2x_2x_3 - 4x_3^2 \\
\text{sub.to} & f_1(\boldsymbol{x}) \equiv -x_1^2 + 5x_2x_3 + 1 \geq 0, \\
& f_2(\boldsymbol{x}) \equiv x_1^2 - 3x_1x_2x_3 + 2x_3 + 2 \geq 0, \\
& f_3(\boldsymbol{x}) \equiv -x_1^2 - x_2^2 - x_3^2 + 1 \geq 0, \\
& x_1(x_1 - 1) = 0 \text{ (0-1 integer)}, \\
& x_2 \geq 0,\ x_3 \geq 0,\ x_2x_3 = 0 \text{ (complementarity)}.
\end{array}
$$

POP: $\min \quad f_0(\boldsymbol{x}) \quad$ sub.to $\quad f_j(\boldsymbol{x}) \geq 0 \; (j = 1, \ldots, m)$.

[1] Lasserre, “Global optimization with polynomials and the problems of moments”, *SIAM J. on Optim.* (2001).
[2] Parrilo, “Semidefinite programming relaxations for semialgebraic problems”, *Math. Prog.* (2003).
[3] Henrion-Lasserre, GloptiPoly.
[4] Prajna-Parachristodoulou-Parrilo, SOSTOOLS.

- [1,3]$\Rightarrow$ a sequence of SDP relaxations — primal approach.
- [2,4]$\Rightarrow$ a sequence of SOS relaxations — dual approach.

(a) Lower bounds for the optimal value.
(b) Convergence to global optimal solutions in theory.
(c) Each relaxed problem can be solved as an SDP; its size gets larger rapidly along “the sequence” as we require a higher accuracy.
(d) Expensive to solve large scale POPs in practice.
$\Rightarrow$ Exploiting Sparsity.

$$\boxed{\text{POP:} \quad \min \quad f_0(\boldsymbol{x}) \quad \text{sub.to} \quad f_j(\boldsymbol{x}) \geq 0 \ (j = 1, \ldots, m).}$$

[1] Lasserre, “Global optimization with polynomials and the problems of moments”, *SIAM J. on Optim.* (2001).

[2] Parrilo, “Semidefinite programming relaxations for semialgebraic problems”, *Math. Prog.* (2003).

[3] Henrion-Lasserre, GloptiPoly.

[4] Prajna-Parachristodoulou-Parrilo, SOSTOOLS.

Exploiting sparsity to solve larger scale problem in practice

[5] Kojima-Kim-Waki, “Sparsity in SOS Polynomials”, *Math. Prog.* (2005).

[6] Waki-Kim-Kojima-Muramatsu, "SOS and SDP Relaxations for POPs with Structured Sparsity", *SIAM J. on Optim* (2006).

[7] Waki-Kim-Kojima-Muramatsu, SparsePOP (2005).

$$\text{POP:} \quad \min \quad f_0(\boldsymbol{x}) \quad \text{sub.to} \quad f_j(\boldsymbol{x}) \geq 0 \; (j = 1, \ldots, m).$$

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[3] Henrion-Lasserre, GloptiPoly.
[4] Prajna-Parachristodoulou-Parrilo, SOSTOOLS.

Extension to polynomial SDP and SOCP

[8] Kojima, “SOS relaxations of poly. SDPs” (2003).
[9] Hol-Scherer,“SOS relaxations of poly. SDPs” (2004).
[10] Henrion-Lasserre, “Convergent relaxations of poly. mat. inequalities & static output feedback ” (2006).
[11] Kojima-Muramatsu, "An Extension of SOS Relaxations to POPs over Symmetric Cones ", to appear in *Math. Prog.*

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$$\boxed{\text{POP:} \quad \min \quad f_0(\boldsymbol{x}) \quad \text{sub.to} \quad f_j(\boldsymbol{x}) \geq 0 \ (j = 1, \ldots, m).}$$

How do we exploit sparsity in POP?

$\Downarrow$

The answer depends on which methods we use to solve POP.

POP

$\Downarrow$ SDP relaxation (Lasserre 2001)

SDP $\Leftarrow$ Primal-Dual IPM (Interior-Point Method)

We will assume a structured sparsity (correlative sparsity):
(a) The size of SDP gets smaller.
(b) SDP satisfies "a similar structured sparsity" under which Primal-Dual IPM works efficiently.

- Characterized in terms of a sparse Cholesky factorization
- Characterized in terms of a sparse chordal graph structure
- Useful to solve large-scale sparse POPs in practice

POP min. $f_0(\boldsymbol{x})$ s.t. $f_j(\boldsymbol{x}) \geq 0$ or $= 0$ $(j = 1, \ldots, m)$.

Example: $f_0(\boldsymbol{x}) = \sum_{k=1}^{n} (-x_k^2)$
$f_j(\boldsymbol{x}) = 1 - x_j^2 - 2x_{j+1}^2 - x_n^2$ $(j = 1, \ldots, n-1)$.

$\boldsymbol{H} f_0(\boldsymbol{x})$: the $n \times n$ Hes. mat. of $f_0(\boldsymbol{x})$,

$\boldsymbol{J} \boldsymbol{f}_*(\boldsymbol{x})$: the $m \times n$ Jacob. mat. of $\boldsymbol{f}_*(\boldsymbol{x}) = (f_1(\boldsymbol{x}), \ldots, f_m(\boldsymbol{x}))^T$,

$\boldsymbol{R}$: the csp matrix, the $n \times n$ sparsiity pattern matrix of
$\boldsymbol{I} + \boldsymbol{H} f_0(\boldsymbol{x}) + \boldsymbol{J} \boldsymbol{f}_*(\boldsymbol{x})^T \boldsymbol{J} \boldsymbol{f}_*(\boldsymbol{x})$ (no cancellation in '$+$').

$[\boldsymbol{J} \boldsymbol{f}_*(\boldsymbol{x})^T \boldsymbol{J} \boldsymbol{f}_*(\boldsymbol{x})]_{ij} \neq 0$ iff x_i and x_j are in a common constraint.

Example with n = 6:

$$\text{the csp matrix } \boldsymbol{R} = \begin{pmatrix} \star & \star & 0 & 0 & 0 & \star \\ \star & \star & \star & 0 & 0 & \star \\ 0 & \star & \star & \star & 0 & \star \\ 0 & 0 & \star & \star & \star & \star \\ 0 & 0 & 0 & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \end{pmatrix}$$

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POP : c-sparse (correlatively sparse) $\Leftrightarrow$ The $n \times n$ csp matrix $\boldsymbol{R} = (R_{ij})$ allows a symbolic sparse Cholesky factorization (under a row & col. ordering like a symmetric min. deg. ordering).

POP min. $f_0(\boldsymbol{x})$ s.t. $f_j(\boldsymbol{x}) \geq 0$ or $= 0$ $(j = 1, \ldots, m)$.

Example: $f_0(\boldsymbol{x}) = \sum_{k=1}^{n} (-x_k^2)$ ——— c-sparse
$f_j(\boldsymbol{x}) = 1 - x_j^2 - 2x_{j+1}^2 - x_n^2$ $(j = 1, \ldots, n-1)$.

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Sparse (SDP) relaxation = Lasserre (2001) + c-sparsity

POP min. $f_0(\boldsymbol{x})$ s.t. $f_j(\boldsymbol{x}) \geq 0$ or $= 0$ $(j = 1, \ldots, m)$, c-sparse.

$$\Downarrow$$

A sequence of c-sparse SDP relaxation problems depending on the relaxation order $r = 1, 2, \ldots$;

- (a) Under a moderate assumption, opt. sol. of SDP $\rightarrow$ opt sol. of POP as $r \rightarrow \infty$.
- (b) $r = \lceil$"the max. deg. of poly. in POP"$/2\rceil + 0 \sim 3$ is usually large enough to attain opt sol. of POP in practice.
- (c) Such an r is unknown in theory except $\exists$ special cases.
- (d) Additional method for all opt. sol. of POP, but expensive.
- (e) The size of SDP increases rapidly as $r \rightarrow \infty$.

POP min. $f_0(\boldsymbol{x})$ s.t. $f_j(\boldsymbol{x}) \geq 0$ or $= 0$ $(j = 1, \ldots, m)$, c-sparse.

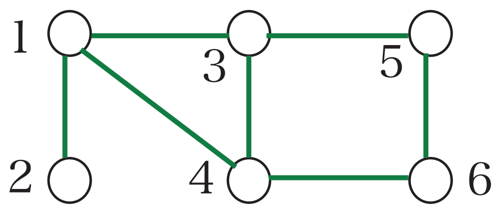
$\boldsymbol{R}$: the csp matrix, the $n \times n$ sparsiity pattern matrix of $\boldsymbol{I} + \boldsymbol{H} f_0(\boldsymbol{x}) + \boldsymbol{J}\boldsymbol{f}_*(\boldsymbol{x})^T \boldsymbol{J}\boldsymbol{f}_*(\boldsymbol{x})$ (no cancellation in '$+$').

POP : c-sparse (correlatively sparse) $\Leftrightarrow$ The $n \times n$ csp matrix $\boldsymbol{R} = (R_{ij})$ allows a symbolic sparse Cholesky factorization.

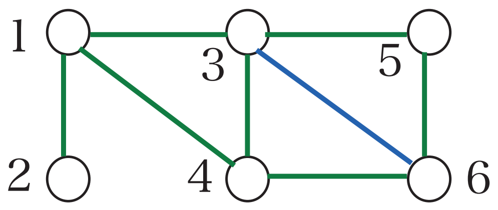
$$E = \{\{i, j\} \in N \times N : \boldsymbol{R}_{ij} = \star,\ i \neq j\} \Updownarrow$$

POP : c-sparse (correlatively sparse) $\Leftrightarrow$ The csp graph $G(N, E)$ has a sparse chordal extension $G(N, \bar{E})$; $E \subseteq \bar{E}$.

$G(N, E)$: not chordal



$G(N, \bar{E})$: chordal



$\forall$ cycle having more than 3 edges has a chord.

- The added edge $\{3, 6\}$ is corresponding to a fill-in.
- The maximal cliques $= \{1, 2\}, \{1, 3, 4\}, \{3, 4, 6\}, \{3, 5, 6\}$.

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POP : c-sparse (correlatively sparse) $\Leftrightarrow$ The $n \times n$ csp matrix $\boldsymbol{R} = (R_{ij})$ allows a symbolic sparse Cholesky factorization.

Two steps to derive a sparse SDP relaxation of POP

(a) Using the max. cliques J_q $(q \in L)$ of $G(N, \bar{E})$, convert POP into an equivalent poly.SDP with the csp graph $G(N, \bar{E})$.

(b) Linearize poly.SDP $\Rightarrow$ SDP with a similar sparsity to poly.SDP.

$\forall$ int. $s \geq 0$, $\boldsymbol{u}_s(x_i : i \in J_q)$: the col. vect. of monomials with deg. at most s in var. x_i $(i \in J_q)$; *e.g.*, if $J_q = \{1, 4\}$,

$$\begin{aligned}
s = 0 &\Rightarrow \boldsymbol{u}_0(x_i : i \in J_q) = 1, \\
s = 1 &\Rightarrow \boldsymbol{u}_1(x_i : i \in J_q) = (1, x_1, x_4)^T, \\
s = 3 &\Rightarrow \boldsymbol{u}_3(x_i : i \in J_q) = (1, x_1, x_4, x_1^2, x_1x_4, x_4^2, x_1^3, x_1^2x_4, x_1x_4^2, x_4^3)^T.
\end{aligned}$$

POP min. $f_0(\boldsymbol{x})$ s.t. $f_j(\boldsymbol{x}) \geq 0$ or $= 0$ $(j = 1, \ldots, m)$, c-sparse.

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$$s = 1 \Rightarrow \boldsymbol{u}_1(x_i : i \in J_q) = (1, x_1, x_4)^T,$$

$$s = 1 \Rightarrow \boldsymbol{u}_1(x_i : i \in J_q)\boldsymbol{u}_1(x_i : i \in J_q)^T = \begin{pmatrix} 1 & x_1 & x_4 \\ x_1 & x_1^2 & x_1 x_4 \\ x_4 & x_1 x_4 & x_4^2 \end{pmatrix}$$

Example of Sparse SDP Relaxation

POP: min $\sum_{i=1}^{4}(-x_i^3)$ s.t. $-a_i \times x_i^2 - x_4^2 + 1 \geq 0 \ (i = 1, 2, 3)$.

$$\text{csp matrix } \boldsymbol{R} = \begin{pmatrix} \star & 0 & 0 & \star \\ 0 & \star & 0 & \star \\ 0 & 0 & \star & \star \\ \star & \star & \star & \star \end{pmatrix}$$

No fill-in in the Cholesky factorization $\Rightarrow$ c-sparse.

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$\Updownarrow$ with the relaxation order $r = 2 \geq r_0 = \lceil 3/2 \rceil = 2$

poly.SDP:

min $\sum_{i=1}^{4}(-x_i^3)$

s.t. $(-a_i \times x_i^2 - x_4^2 + 1)(1, x_i, x_4)^T(1, x_i, x_4) \succeq \boldsymbol{O} \quad i = 1, 2, 3,$

$(1, x_j, x_4, x_j^2, x_j x_4, x_4^2)^T(1, x_j, x_4, x_j^2, x_j x_4, x_4^2) \succeq \boldsymbol{O} \ j = 1, 2, 3.$

$$\text{csp matrix } \boldsymbol{R} = \begin{pmatrix} \star & 0 & 0 & \star \\ 0 & \star & 0 & \star \\ 0 & 0 & \star & \star \\ \star & \star & \star & \star \end{pmatrix}$$

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$(1, x_j, x_4, x_j^2, x_j x_4, x_4^2)^T(1, x_j, x_4, x_j^2, x_j x_4, x_4^2) \succeq \boldsymbol{O} \ j = 1,2,3.$

Represent poly.SDP as

min $\sum_{\boldsymbol{\alpha}\in\mathcal{A}_0} g_0(\boldsymbol{\alpha})\boldsymbol{x}^{\boldsymbol{\alpha}}$ s.t. $\sum_{\boldsymbol{\alpha}\in\mathcal{A}_j} \boldsymbol{G}_j(\boldsymbol{\alpha})\boldsymbol{x}^{\boldsymbol{\alpha}} \succeq \boldsymbol{O} \ j = 1,\ldots,6,$

where $\mathcal{A}_j \subset \mathbb{Z}_+^4$ and $\boldsymbol{x}^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}x_4^{\alpha_4}$; $\boldsymbol{x}^{(1,2,1,0)} = x_1 x_2^2 x_3$.

$\Downarrow$ Linearize by replacing each $\boldsymbol{x}^{\boldsymbol{\alpha}}$ by an indep. var. $y_{\boldsymbol{\alpha}}$; $\boldsymbol{x}^0$ by 1

SDP min $\displaystyle\sum_{\boldsymbol{\alpha}\in\mathcal{A}_0} g_0(\boldsymbol{\alpha})y_{\boldsymbol{\alpha}}$ s.t. $\displaystyle\sum_{\boldsymbol{\alpha}\in\mathcal{A}_j} \boldsymbol{G}_j(\boldsymbol{\alpha})y_{\boldsymbol{\alpha}} \succeq \boldsymbol{O} \ j = 1,\ldots,6,$

which forms an SDP relaxation of POP.

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Numerical result on unconstrained optimization

Unconstrained POP: min. $f(\boldsymbol{x})$, $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

Broyden tridiagonal function with min.val.$= 0$

$f(\boldsymbol{x}) = \sum_{i=1}^{n} \left((3-2x_i)x_i - x_{i-1} - 2x_{i+1} + 1\right)^2, \ x_0 = x_{n+1} = 0.$

Generalized Rosenbrock function with min.val.$= 0$

$f(\boldsymbol{x}) = \sum_{i=2}^{n} \left(100(x_i - x_{i-1}^2)^2 + (1-x_i)^2\right), \ x_1 \geq 0.$.

		B. tridiagonal		G. Rosenbrock	
n	r	approx.opt.val	cpu	approx.opt.val	cpu
600	2	1.0e-7	9.3	3.9e-7	3.4
800	2	2.2e-7	12.6	2.1e-7	5.1
1000	2	2.6e-7	16.0	4.5e-7	5.9

An optimal control problem from Coleman et al. 1995

$$\left.\begin{array}{ll} \text{min} & \dfrac{1}{M}\displaystyle\sum_{i=1}^{M-1}(y_i^2 + x_i^2) \\ \text{s.t.} & y_{i+1} = y_i + \dfrac{1}{M}(y_i^2 - x_i), \quad (i = 1, \ldots, M-1), \quad y_1 = 1. \end{array}\right\}$$

Numerical results on sparse relaxation

M	r	# of variables	ϵ_{obj}	ϵ_{feas}	cpu
600	2	1198	3.4e-08	2.2e-10	3.4
800	2	1598	5.9e-08	1.6e-10	3.8
1000	2	1998	6.3e-08	2.7e-10	5.0

$$\epsilon_{\text{obj}} = \frac{|\text{the lower bd. for opt. val.} - \text{the approx. opt. val.}|}{\max\{1, |\text{the lower bd. for opt. val.}|\}},$$

ϵ_{feas} = the maximum error in the equality constraints,

cpu : cpu time in sec. to solve an SDP relaxation problem.

alkyl.gms : a benchmark problem from globalib

$$
\begin{array}{ll}
\text{min} & -6.3x_5x_8 + 5.04x_2 + 0.35x_3 + x_4 + 3.36x_6 \\
\text{sub.to} & -0.820x_2 + x_5 - 0.820x_6 = 0, \\
& 0.98x_4 - x_7(0.01x_5x_{10} + x_4) = 0, \\
& -x_2x_9 + 10x_3 + x_6 = 0, \\
& x_5x_{12} - x_2(1.12 + 0.132x_9 - 0.0067x_9^2) = 0, \\
& x_8x_{13} - 0.01x_9(1.098 - 0.038x_9) - 0.325x_7 = 0.574, \\
& x_{10}x_{14} + 22.2x_{11} = 35.82,\ x_1x_{11} - 3x_8 = -1.33, \\
& \text{lbd}_i \leq x_i \leq \text{ubd}_i \ (i = 1, 2, \ldots, 14).
\end{array}
$$

	Sparse			Dense (Lasserre)		
r	ϵ_{obj}	ϵ_{feas}	cpu	ϵ_{obj}	ϵ_{feas}	cpu
2	1.0e-02	7.1e-01	1.8	7.2e-3	4.3e-2	14.4
3	5.6e-10	2.0e-08	23.0	out of	memory	

ϵ_{obj} = approx.opt.val. – lower bound for opt.val.
ϵ_{feas} = the maximum error in the equality constraints

Some other benchmark problems from globalib

		Sparse			Dense (Lasserre)	
problem	n	ϵ_{obj}	ϵ_{feas}	cpu	ϵ_{obj}	cpu
ex3_1_1	8	6.3e-9	4.7e-4	5.5	0.7e-8	597.8
st_bpaf1b	10	3.8e-8	2.8e-8	1.0	4.6e-9	1.7
st_e07	10	0.0	8.1e-5	0.4	0.0	3.0
ex2_1_3	13	5.1e-09	3.5e-9	0.5	1.6e-9	7.7
ex9_1_1	13	0.0	4.5e-6	1.5	0.0	7.7
ex9_2_3	16	0.0	5.7e-6	2.3	0.0	49.7
ex2_1_8	24	1.0e-5	0.0	304.6	3.4e-6	1946.6
ex5_2_2	9	1.0e-2	3.2e+1	1.8	1.6e-5	2.6

- ex5_2_2 — Dense is better.
- Sparse attains approx. opt. solutions with the same quality as Dense except ex5_2_2.
- Sparse is much faster than Dense in large dim. cases.

Discretization of Mimura's ODE with 2 unknowns $u,\ v : [0,5] \to \mathbb{R}$

$$
\begin{aligned}
u_{xx} &= -(\delta_1/9)(35 + 16u - u^2)u + (\delta_1)(kuv), \\
v_{xx} &= (\delta_2)((1 + (2/5)v)v - kuv), \\
& \underline{u_x(0) = u_x(5) = v_x(0) = v_x(5) = 0,}
\end{aligned}
$$

where $k = 1$, $\delta_1 = 20$ and $\delta_2 = 1/4$. Discretize:

$x_i = i\Delta x \ (i = 0, 1, 2, \dots),\ u_x(x_i) \approx (u(x_{i+1}) - u(x_{i-1}))/(2\Delta x)$.

Discretized system of polynomials with $\Delta x = 1$:

$$
\begin{aligned}
f_1(\boldsymbol{u},\boldsymbol{v}) &= 76.8u_1 + u_3 + 35.6u_1^2 - 20.0u_1v_1 - 2.22u_2^3, \\
f_2(\boldsymbol{u},\boldsymbol{v}) &= -1.25v_1 + v_2 + 0.25u_1v_1 - 0.1v_1^2, \\
f_3(\boldsymbol{u},\boldsymbol{v}) &= u_1 + 75.8u_2 + u_3 + 35.6u_2^2 - 20.0u_2v_2 - 2.22u_2^3, \\
f_4(\boldsymbol{u},\boldsymbol{v}) &= v_1 - 2.25v_2 + v_3 + 0.25u_2v_2 - 0.1v_2^2, \\
f_5(\boldsymbol{u},\boldsymbol{v}) &= u_2 + 75.8u_3 + u_4 + 35.6u_3^2 - 20.0u_3v_3 - 2.22u_3^2, \\
f_6(\boldsymbol{u},\boldsymbol{v}) &= v_2 - 2.25v_3 + v_4 + 0.25u_3v_3 - 0.1v_3^2, \\
f_7(\boldsymbol{u},\boldsymbol{v}) &= u_3 + 76.8u_4 + 35.6u_4^2 - 20.0u_4v_4 - 2.22u_4^3, \\
f_8(\boldsymbol{u},\boldsymbol{v}) &= v_3 - 1.25v_4 + 0.25u_4v_4 - 0.1v_4^2.
\end{aligned}
$$

Here $u_i = u(x_i)$, $v_i = v(x_i)$ $(i = 0, 1, 2, 3, 4, 5)$,
$u_0 = u_1$, $u_5 = u_4$, $v_0 = v_1$ and $v_5 = v_4$.

$\Rightarrow$ c-sparse

Discretization of Mimura's ODE with 2 unknowns $u, \ v : [0,5] \to \mathbb{R}$

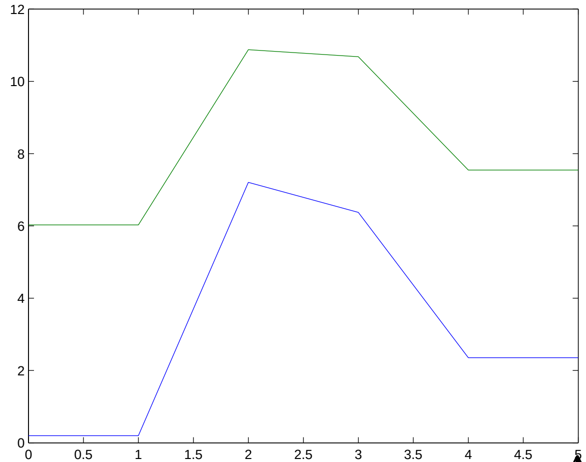
$$u_{xx} = -(\delta_1/9)(35 + 16u - u^2)u + (\delta_1)(kuv),$$
$$v_{xx} = (\delta_2)((1 + (2/5)v)v - kuv),$$
$$u_x(0) = u_x(5) = v_x(0) = v_x(5) = 0,$$

where $k = 1$, $\delta_1 = 20$ and $\delta_2 = 1/4$. Discretize:
$x_i = i\Delta x \ (i = 0, 1, 2, \dots), \ u_x(x_i) \approx (u(x_{i+1}) - u(x_{i-1}))/(2\Delta x)$.

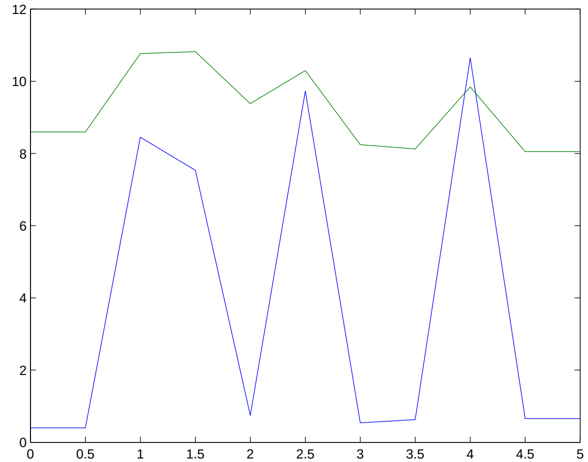
- Numerical results on SparsePOP

Δx	n	obj.funct.	r	$\boldsymbol{A}$ in SeDuMi	cpu
1.0	8	$\sum r_i u(x_i) \uparrow$	3	[1,084, 18,732]	11.3
0.5	18	$\sum r_i u(x_i) \uparrow$	3	[3,025, 48,285]	57.8

Here $r_i \in (0,1)$: random numbers.



$\Delta x = 1.0$



$\Delta x = 0.5$

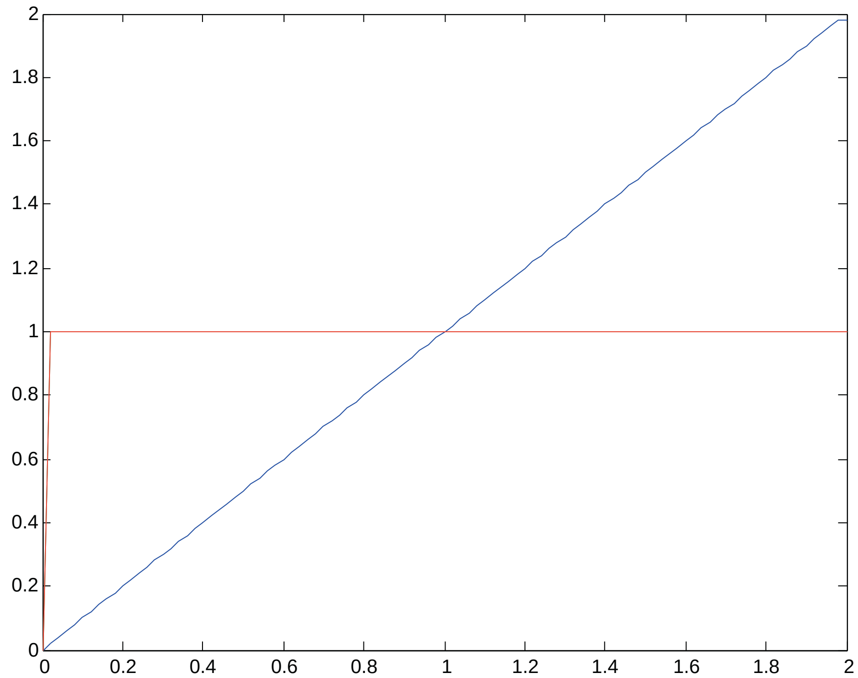
Discretization of DAE with 3 unknowns $y_1, y_2, y_3 : [0, 2] \to \mathbb{R}$

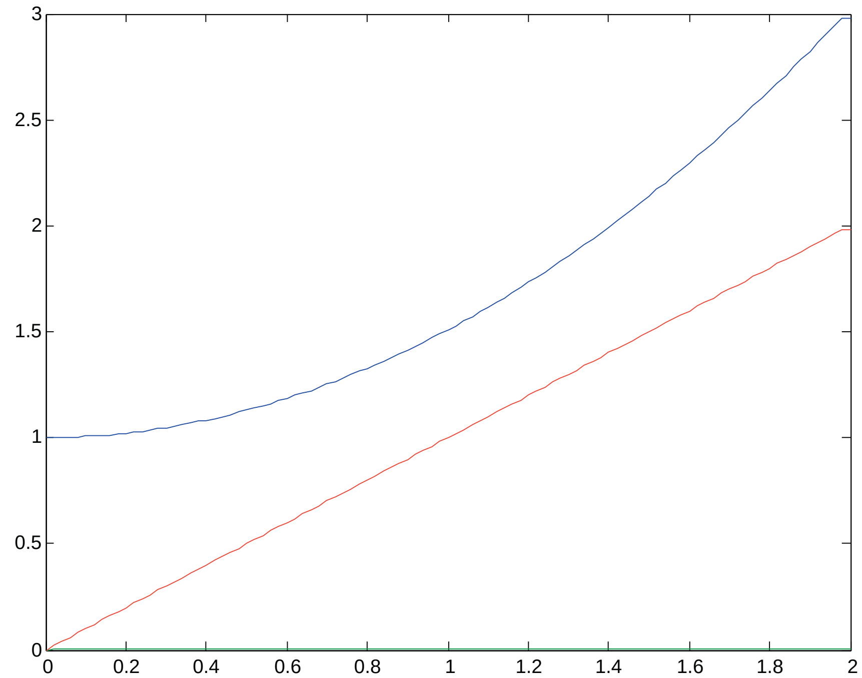
$y_1' = y_3,\ 0 = y_2(1 - y_2),\ 0 = y_1 y_2 + y_3(1 - y_2) - t,\ y_1(0) = y_1^0.$

2 solutions : $y(t) = (t, 1, 1)$ and $y(t) = (y_1^0 + t^2, 0, t)$.

- Numerical results on SparsePOP — c-sparse

y_1^0	Δt	n	obj.funct.	r	$\boldsymbol{A}$ in SeDuMi	cpu
0	0.02	297	$\sum y_2(t_i) \uparrow$	2	[3,557, 25,413]	30.9
1	0.02	297	$\sum y_1(t_i) \uparrow$	2	[3,557, 25,413]	33.9





Outline

1. POPs (Polynomial Optimization Problems)
2. How do we formulate structured sparsity?
3. Sparse SDP relaxation of POPs
4. Numerical results

5. Concluding remarks

- Lasserre's (dense) relaxation
  — Theoretical convergence but expensive in practice.
- Sparse relaxation (Waki-Kim-Kojima-Muramatsu)
  = Lasserre's (dense) relaxation + sparsity
  — Very powerful in practice
  and theoretical convergence (Lasserre)
- There remain many issues to be studied further.
  - Exploiting sparsity.
  - Large-scale SDPs.
  - Numerical difficulty in solving SDP relaxations of POPs.

Thank you!