Conversion Methods for Large Scale SDPs and Their Applications to Polynomial Optimization Problems

Workshop: Advances in Mathematical Modeling

and Computational Algorithms in Information Processing

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Contents

1. Introduction
   — Semidefinite Programs (SDPs) and their conversion —
2. Conversion methods
   - 2-1. Aggregated sparsity and positive semidefinite matrix completion
   - 2-2. Conversion to a Primal form SDP with small mat. variables
   - 2-3. Conversion to an LMI form SDP with small mat. variables
3. Applications to SDP relaxation
   - 3-1. Sensor network localization problems
   - 3-2. Polynomial optimization problems
   - 3-3. Polynomial SDPs
4. Concluding remarks

Contents

Equality standard form SDP:

min $\boldsymbol{A}_0 \bullet \boldsymbol{X}$ sub.to $\boldsymbol{A}_p \bullet \boldsymbol{X} = b_p \ (p = 1, \ldots, m), \ \mathcal{S}^n \ni \boldsymbol{X} \succeq \boldsymbol{O}$

$\boldsymbol{A}_p \in \mathcal{S}^n$ the linear space of $n \times n$ symmetric matrices

with the inner product $\boldsymbol{A}_p \bullet \boldsymbol{X} = \sum_{i,\ j} [A_p]_{ij} X_{ij}$.

$b_p \in \mathbb{R}, \ \boldsymbol{X} \succeq \boldsymbol{O} \ \Leftrightarrow \ \boldsymbol{X} \in \mathcal{S}^n$ is positive semidefinite.

Lots of Applications to Various Problems

- Systems and control theory — Linear Matrix Inequality
- SDP relaxations of combinatorial and nonconvex problems
  - Max cut and max clique problems
  - Quadratic assignment problems
  - Polynomial optimization problems — later
  - Polynomial semidefinite programs — later
- Robust optimization
- Quantum chemistry
- Moment problems (applied probability)
- Sensor network localization problem — later
- . . .

Equality standard form SDP:

min $\boldsymbol{A}_0 \bullet \boldsymbol{X}$ sub.to $\boldsymbol{A}_p \bullet \boldsymbol{X} = b_p \ (p = 1, \ldots, m), \ \mathcal{S}^n \ni \boldsymbol{X} \succeq \boldsymbol{O}$

SDP can be large-scale easily

- $n \times n$ mat. variable $\boldsymbol{X}$ involves $n(n+1)/2$ real variables; $n = 2000 \Rightarrow n(n+1)/2 \approx 2$ million
- m linear equality constraints or m $\boldsymbol{A}_p$'s $\in \mathcal{S}^n$

◊ How can we solve a larger scale SDP?

(a) Use more powerful computer system such as clusters and grids of computers — parallel computation.

(b) Develop new numerical methods for SDPs.

(c) Improve primal-dual interior-point methods.

(d) Convert a large sparse SDP to an SDP which existing pdipms can solve efficiently:

- multiple but small size mat. variables.
- a sparse Schur complement mat. (a coef. mat. of a sys. of equations solved at ∀ iteration of the pdipm).

Outline of conversion methods

structured sparsity used	a large scale and structured sparse SDP	technique
aggregated sparsity	$\Downarrow$	positive semidefinite mat. completion
	an SDP with small SDP cones and shared variables among SDP cones	
	$\Downarrow$ $\Downarrow$	conversion to Equality form SDP or conversion to LMI form SDP
	an SDP with small mat. variables (*i.e.*, small SDP cones)	

Contents

1. Introduction
   — Semidefinite Programs (SDPs) and their conversion —
2. Conversion methods
   - 2-1. Aggregated sparsity and positive semidefinite matrix completion (Fukuda et al. ’01, Nakata et al. ’03)
   - 2-2. Conversion to a Primal form SDP with small mat. variables (Fukuda et al. ’01, Nakata et al. ’03)
   - 2-3. Conversion to an LMI form SDP with small mat. variables
3. Applications to SDP relaxation
   - 3-1. Sensor network localization problems
   - 3-2. Polynomial optimization problems
   - 3-3. Polynomial SDPs
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Equality standard form SDP:

min $\boldsymbol{A}_0 \bullet \boldsymbol{X}$ sub.to $\boldsymbol{A}_p \bullet \boldsymbol{X} = b_p \ (p = 1, \ldots, m), \ \mathcal{S}^n \ni \boldsymbol{X} \succeq \boldsymbol{O}$

$$E_* \ = \ \{(i,j) : i = j \text{ or } [A_p]_{ij} \neq 0 \text{ for } \exists p = 0, \ldots, m\}$$

$\boldsymbol{A}_*$: $n \times n$ aggregated sparsity pattern mat.

$$[A_*]_{ij} \ = \ \star \text{ if } (i,j) \in E_* \text{ and } 0 \text{ oterrwise}$$

SDP : a-sparse if $\boldsymbol{A}_*$ allows a sparse Cholesky factorization

Two typical cases: 1. bandwidth along diagonal

$$\boldsymbol{A}_* = \begin{pmatrix} \star & \star & 0 & 0 & 0 \\ \star & \star & \star & 0 & 0 \\ 0 & \star & \star & \star & 0 \\ 0 & 0 & \star & \star & \star \\ 0 & 0 & 0 & \star & \star \end{pmatrix}$$

min $\displaystyle\sum_{(i,j)\in E_*} [A_0]_{ij} X_{ij}$

sub.to $\displaystyle\sum_{(i,j)\in E_*} [A_p]_{ij} X_{ij} = b_p \ (\forall p)$

$$\begin{pmatrix} X_{qq} & X_{q,q+1} \\ X_{q+1,q} & X_{q+q,q+1} \end{pmatrix} \succeq \boldsymbol{O}$$

$(q = 1, \ldots, n-1)$.

SDP = SDP with shared variables among small SDP cones

Each $\star$ can be a block matrix.

Equality standard form SDP:

min $\boldsymbol{A}_0 \bullet \boldsymbol{X}$ sub.to $\boldsymbol{A}_p \bullet \boldsymbol{X} = b_p \ (p = 1, \ldots, m), \ \mathcal{S}^n \ni \boldsymbol{X} \succeq \boldsymbol{O}$

$$E_* \ = \ \{(i,j) : i = j \ \text{ or } [A_p]_{ij} \neq 0 \text{ for } \exists p = 0, \ldots, m\}$$

$\boldsymbol{A}_*$: $n \times n$ aggregated sparsity pattern mat.

$$[A_*]_{ij} \ = \ \star \ \text{ if } (i,j) \in E_* \ \text{ and } 0 \text{ oterrwise}$$

SDP : a-sparse if $\boldsymbol{A}_*$ allows a sparse Cholesky factorization

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Two typical cases: 2. arrow $\searrow$

$$\boldsymbol{A}_* = \begin{pmatrix} \star & 0 & 0 & 0 & \star \\ 0 & \star & 0 & 0 & \star \\ 0 & 0 & \star & 0 & \star \\ 0 & 0 & 0 & \star & \star \\ \star & \star & \star & \star & \star \end{pmatrix}$$

min $\displaystyle\sum_{(i,j)\in E_*} [A_0]_{ij} X_{ij}$

sub.to $\displaystyle\sum_{(i,j)\in E_*} [A_p]_{ij} X_{ij} = b_p \ (\forall p)$

$$\begin{pmatrix} X_{qq} & X_{qn} \\ X_{nq} & X_{nn} \end{pmatrix} \succeq \boldsymbol{O}$$

$(q = 1, \ldots, n-1)$.

SDP = SDP with shared variables among small SDP cones

Each $\star$ can be a block matrix.

Equality standard form SDP:

min $A_0 \bullet X$ sub.to $A_p \bullet X = b_p \ (p = 1, \ldots, m), \ \mathcal{S}^n \ni X \succeq O$

$$E_* = \{(i,j) : i = j \text{ or } [A_p]_{ij} \neq 0 \text{ for } \exists p = 0, \ldots, m\}$$

A_* : $n \times n$ aggregated sparsity pattern mat.

$$[A_*]_{ij} = \star \text{ if } (i,j) \in E_* \text{ and } 0 \text{ oterrwise}$$

SDP : a-sparse if A_* allows a sparse Cholesky factorization

$\Downarrow$ positive semidefinite matrix completion

$\exists C_1, \ldots, C_\ell \subset N = \{1, 2, \ldots, n\}, \ \ell \leq n;$

SDP $\equiv$ an SDP with shared variables among small SDP cones:

min $\sum_{(i,j) \in E_*} [A_0]_{ij} X_{ij}$

s.t. $\sum_{(i,j) \in E_*} [A_p]_{ij} X_{ij} = b_p \ (\forall p), \ X(C_r) \succeq O \ (r = 1, \ldots, \ell),$

where $X(C_r)$: the submatrix of X consisting of $X_{ij} \ (i,j \in C_r)$.

- To solve SDP, we need to convert it into a standard form SDP $\Rightarrow$ next subject.

Contents

Equality standard form SDP:

min $A_0 \bullet X$ sub.to $A_p \bullet X = b_p \ (p = 1, \ldots, m), \ \mathcal{S}^n \ni X \succeq O$

As an example: $\Downarrow$ aggregated sparsity

min $\sum_{(i,j)\in E_*} [A_0]_{ij} X_{ij}$ sub.to $\sum_{(i,j)\in E_*} [A_p]_{ij} X_{ij} = b_p$ and

$$\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}, \begin{pmatrix} X_{22} & X_{23} & X_{24} \\ X_{32} & X_{33} & X_{34} \\ X_{42} & X_{43} & X_{44} \end{pmatrix}, \begin{pmatrix} X_{33} & X_{34} & X_{35} \\ X_{43} & X_{44} & X_{45} \\ X_{53} & X_{54} & X_{55} \end{pmatrix} \succeq O$$

(an SDP with smaller SDP cones and shared variables) $\Longrightarrow$

Conversion into a standard form SDP to apply IPM — 2 ways

Primal form SDP with small mat. variables:

min "linear obj. in Y^r_{ij}s" sub.to "linear eq. in Y^r_{ij}s" and

$$\begin{pmatrix} Y^1_{11} & Y^1_{12} \\ Y^1_{21} & Y^1_{22} \end{pmatrix}, \begin{pmatrix} Y^2_{11} & Y^2_{12} & Y^2_{13} \\ Y^2_{21} & Y^2_{22} & Y^2_{23} \\ Y^2_{31} & Y^2_{32} & Y^2_{33} \end{pmatrix}, \begin{pmatrix} Y^3_{11} & Y^3_{12} & Y^3_{13} \\ Y^3_{21} & Y^3_{22} & Y^3_{23} \\ Y^3_{31} & Y^3_{32} & Y^3_{33} \end{pmatrix} \succeq O,$$

$$Y^1_{22} = Y^2_{11}, \ Y^2_{22} = Y^3_{11}, \ Y^2_{23} = Y^3_{12}, \ Y^2_{33} = Y^3_{22}.$$

Equality standard form SDP:

min $\boldsymbol{A}_0 \bullet \boldsymbol{X}$ sub.to $\boldsymbol{A}_p \bullet \boldsymbol{X} = b_p \ (p = 1, \ldots, m), \ \mathcal{S}^n \ni \boldsymbol{X} \succeq \boldsymbol{O}$

As an example: $\Downarrow$ aggregated sparsity

min $\sum_{(i,j)\in E_*} [A_0]_{ij} X_{ij}$ sub.to $\sum_{(i,j)\in E_*} [A_p]_{ij} X_{ij} = b_p$ and

$$\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}, \begin{pmatrix} X_{22} & X_{23} & X_{24} \\ X_{32} & X_{33} & X_{34} \\ X_{42} & X_{43} & X_{44} \end{pmatrix}, \begin{pmatrix} X_{33} & X_{34} & X_{35} \\ X_{43} & X_{44} & X_{45} \\ X_{53} & X_{54} & X_{55} \end{pmatrix} \succeq \boldsymbol{O}$$

(an SDP with smaller SDP cones and shared variables) $\Longrightarrow$

Conversion into a standard form SDP to apply IPM — 2 ways

$\Downarrow$ LMI form SDP with small mat. variables — next Section

SDP with small (independent) matrix variables:

min $\sum_{r=1}^{\ell} \boldsymbol{A}_{0r} \bullet \boldsymbol{X}_r$

sub.to $\sum_{r=1}^{\ell} \boldsymbol{A}_{pr} \bullet \boldsymbol{X}_r = b_p \ (p = 1, \ldots, m), \ \boldsymbol{X}_r \succeq \boldsymbol{O} \ (\forall r)$

- Further sparsity "$\boldsymbol{A}_{pr} \equiv \boldsymbol{O}$ for many pairs of p and r" is often satisfied $\Rightarrow$ correlative sparsity

Contents

$$\begin{array}{ll} \text{min} & \sum_{(i,j)\in E_*} [A_0]_{ij} X_{ij} \\ \text{s.t.} & \sum_{(i,j)\in E_*} [A_p]_{ij} X_{ij} = b_p \ (\forall p), \ \boldsymbol{X}(C_r) \succeq \boldsymbol{O} \ (r = 1, \ldots, \ell), \end{array}$$

where $\boldsymbol{X}(C_r)$: the submatrix of $\boldsymbol{X}$ consisting of X_{ij} $(i, j \in C_r)$.

Represent each $\boldsymbol{X}(C_r)$ as

$$\boldsymbol{X}(C_r) = \sum_{i,j\in C_r, i\leq j} \boldsymbol{E}_{ij}(C_r) X_{ij},$$

where $\boldsymbol{E}_{ij}(C_r)$: a sym. mat. with 1 at some one or two elements and 0 elsewhere. For example,

$$\begin{pmatrix} X_{11} & X_{13} \\ X_{31} & X_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} X_{11} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} X_{12} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} X_{33}$$

Then, an LMI form SDP having eq. const.

$$\begin{array}{lll} \text{min} \ \sum_{(i,j)\in E_*} [A_0]_{ij} X_{ij} & \text{sub.to} & \sum_{(i,j)\in E_*} [A_p]_{ij} X_{ij} = b_p \ (\forall p), \\ & & \sum_{i,j\in C_r, i\leq j} \boldsymbol{E}_{ij}(C_r) X_{ij} \succeq \boldsymbol{O} \ (\forall r). \end{array}$$

Review of conversion methods

structured sparsity used	a large scale and structured sparse SDP	technique
aggregated sparsity	⇓	positive semidefinite mat. completion
	an SDP with small SDP cones and shared variables among SDP cones	
	⇓ ⇓	conversion to Equality form SDP or conversion to LMI form SDP
	an SDP with small mat. variables (*i.e.*, small SDP cones)	

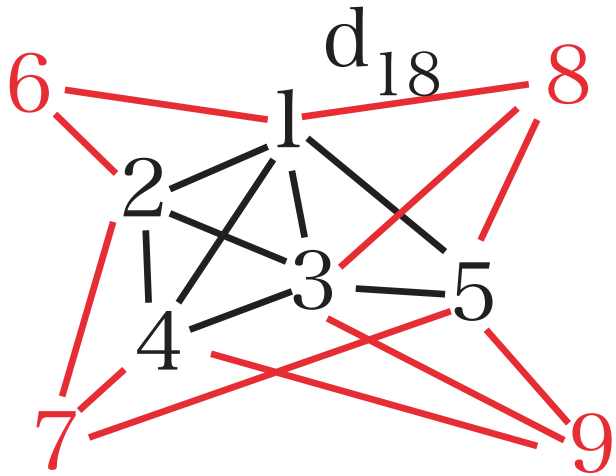
Contents

Sensor network localization problem: Let $s = 2$ or 3.

$$
\begin{array}{rcl}
\boldsymbol{x}^p \in \mathbb{R}^s & : & \text{unknown location of sensors } (p = 1, 2, \ldots, m), \\
\boldsymbol{x}^r = \boldsymbol{a}^r \in \mathbb{R}^s & : & \text{known location of anchors } (r = m+1, \ldots, n), \\
d_{pq} & = & \|\boldsymbol{x}^p - \boldsymbol{x}^q\| + \epsilon_{pq} \text{ — given for } (p, q) \in \mathcal{N}, \\
\mathcal{N} & = & \{(p, q) : \|\boldsymbol{x}^p - \boldsymbol{x}^q\| \leq \rho = \text{a given radio range}\}
\end{array}
$$

Here ϵ_{pq} denotes a noise.

$m = 5, \ n = 9$.
$1, \ldots, 5$: sensors
$6, 7, 8, 9$: anchors



Anchors' positions are known.
A distance is given for $\forall$ edge.
Compute locations of sensors.

$\Rightarrow$ Some nonconvex QOPs

- SDP relaxation — FSDP by Biswas-Ye '06, ESDP by Wang et al '07, ... for $s = 2$.
- SOCP relaxation — Tseng '07 for $s = 2$.
- ...

Numerical results on 4 methods (a), (b), (c) and (d) applied to a sensor network localization problem with

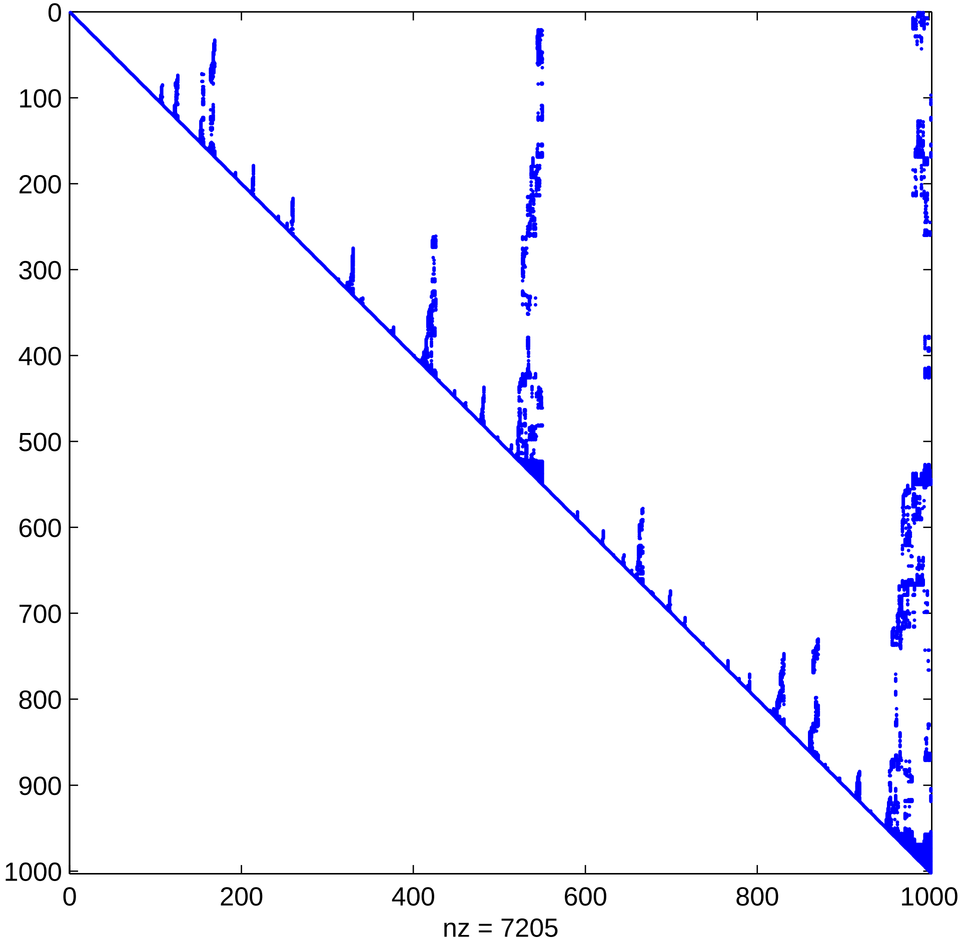
$m =$ the number of sensors dist. randomly in $[0, 1]^2$,
4 anchors located at the corner of $[0, 1]^2$,
$\rho =$ radio distance $= 0.1$, no noise.

(a) FSDP (b) FSDP + Conv. to LMI form SDP, as strong as (a)
(c) FSDP + Conv. to Equality form SDP as strong as (a)

	cpu time for solving SDP by SeDuMi in second		
m	(a)	(b)	(c)
500	389.1	35.0	69.5
1000	3345.2	60.4	178.8
2000		111.1	326.0
4000		182.1	761.0

Cholesky factor of aggregated sparsity pattern $\Rightarrow$ next slide

(a) FSDP — cpu time 3345.2 sec
Cholesky Factor of Aggregated sparsity pattern

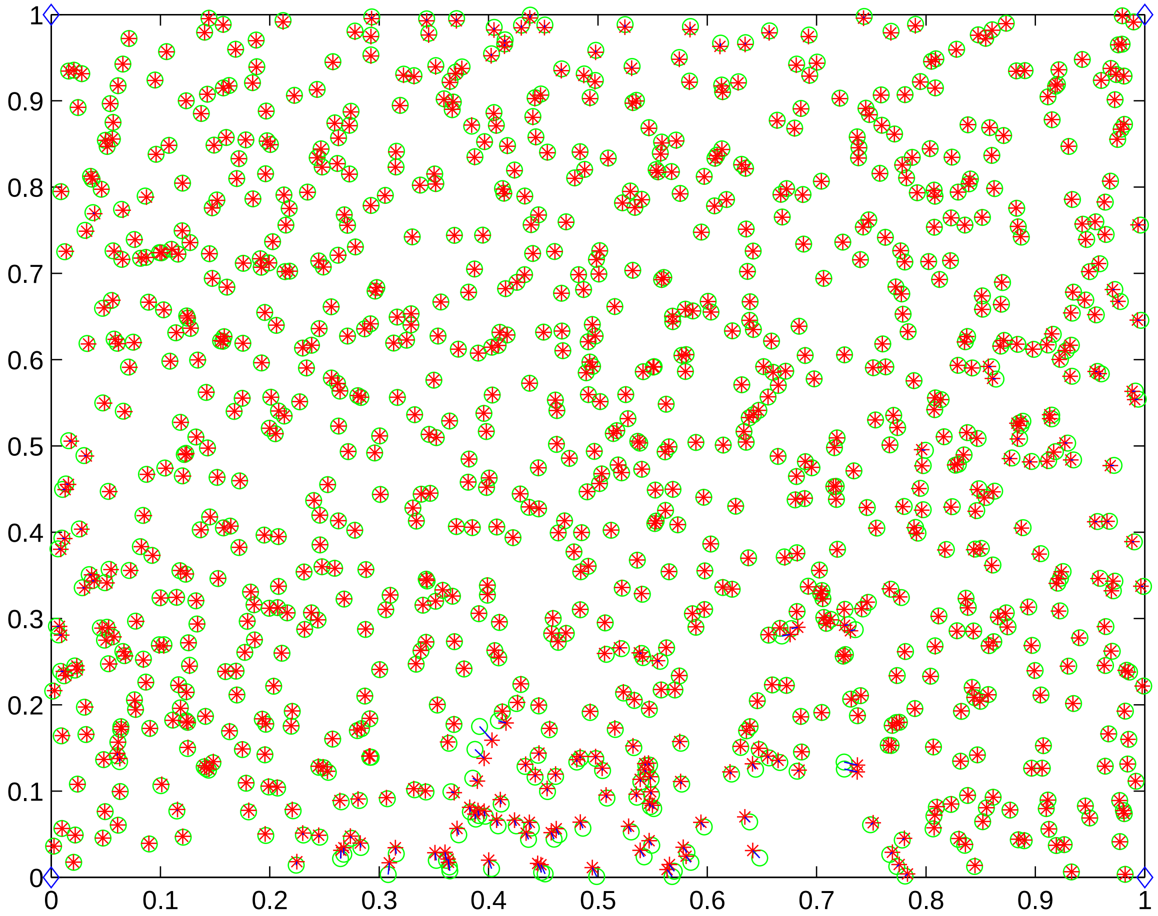


This aggregated sparsity pattern is exploited in

(b) FSDP + Conv. to LMI form SDP — cpu time 60.4 sec

(c) FSDP + Conv. to Equality form SDP — cpu time 178.8 sec

(b) FSDP + Conv. to LMI form SDP — cpu time 60.4 sec
(c) FSDP + Conv. to Equality form SDP — cpu time 178.8 sec



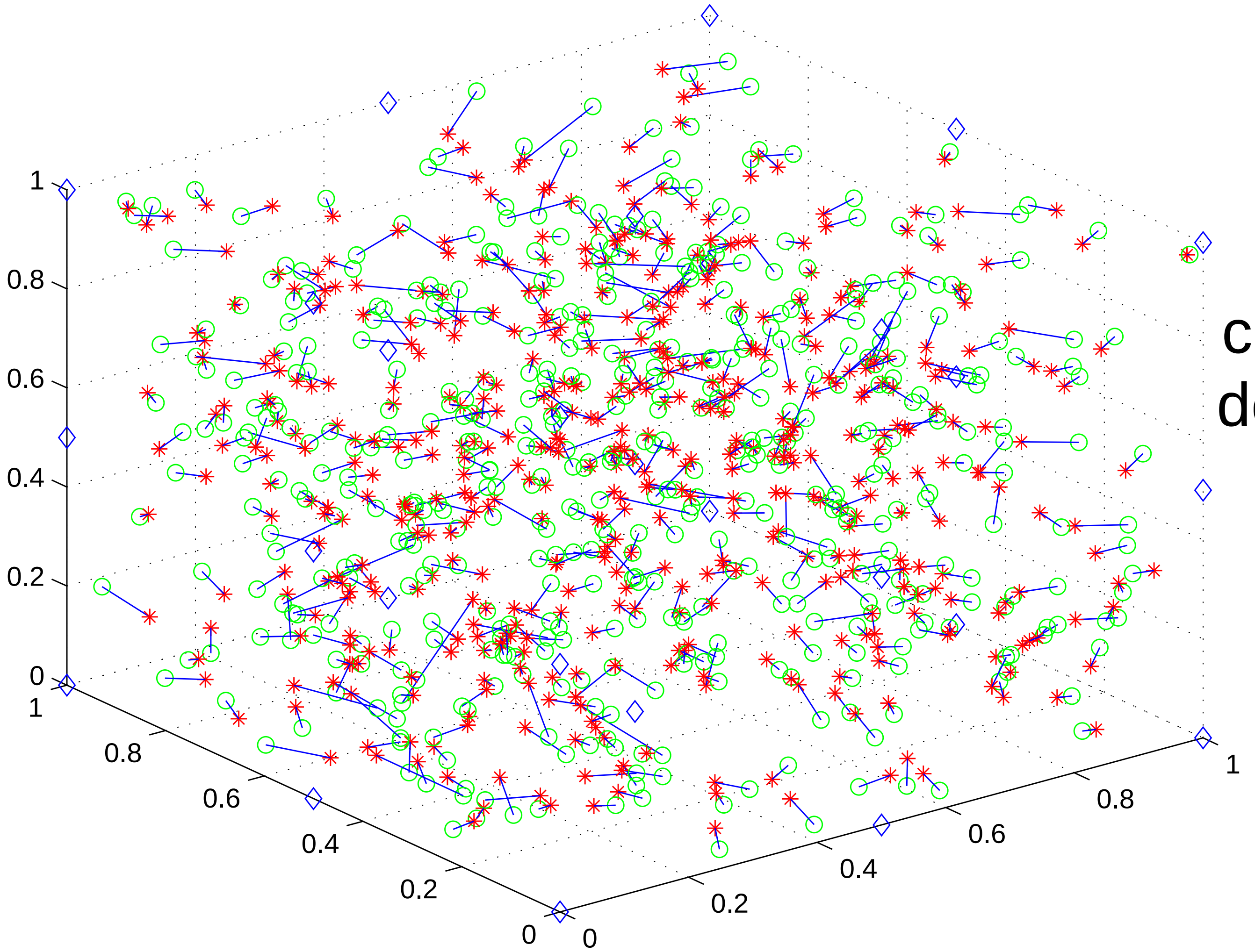
anchor : ◇
true : ○
computed : ∗
deviation : —

3 dim, 500 sensors, 27 anchors, r.range = 0.3, noise $\leftarrow$ N(0,0.1);

(estimated dist.) $\hat{d}_{pq} = (1+\epsilon_{pq})d_{pq}$ (true unknown dist.),

$\epsilon_{pq} \leftarrow N(0, 0.1)$

(b) FSDP + Conv. to LMI form SDP



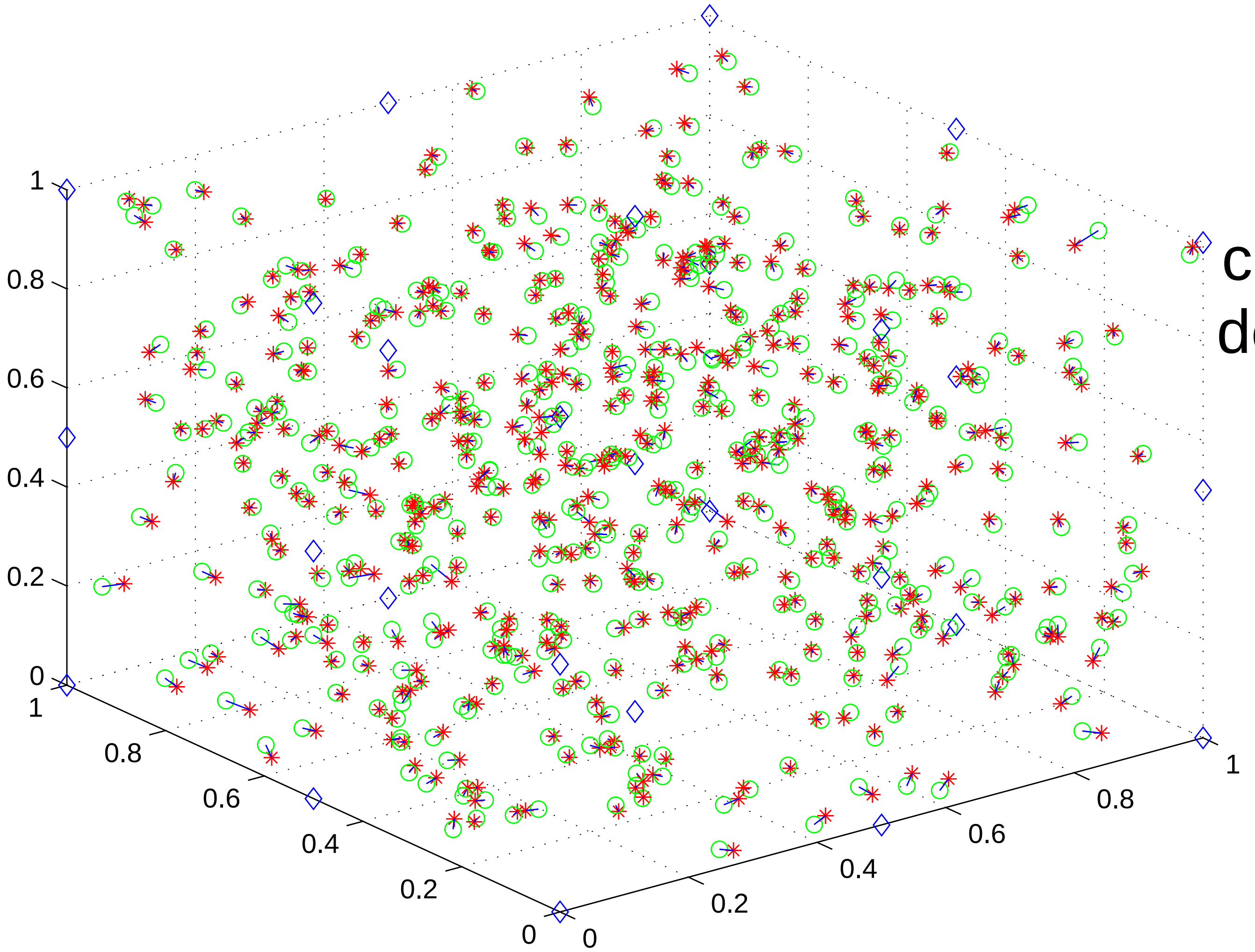
anchor : ◇
true : ○
computed : *
deviation : —
24.2 sec.

3 dim, 500 sensors, 27 anchors, r.range = 0.3, noise $\leftarrow$ N(0,0.1);

(estimated dist.) $\hat{d}_{pq} = (1 + \epsilon_{pq}) d_{pq}$ (true unknown dist.),

$\epsilon_{pq} \leftarrow N(0, 0.1)$

(b) FSDP + Conv. to LMI form SDP + Gradient method



anchor : ◊
true : ○
computed : *
deviation : —
24.2 sec.
+ 8.4 sec.

Contents

POP (Polynomial Optimization Problem)

$$\min\ f_0(\boldsymbol{x}) \text{ sub.to } f_i(\boldsymbol{x}) \geq 0 \ (i = 1, 2, \ldots, m).$$

Here $f_p(\boldsymbol{x})$ denotes a polynomial in $\boldsymbol{x} = (x_1, \ldots, x_n)$.

(a) Apply SDP relaxation to POP $\Rightarrow$ SDP
— SparsePOP(MATLAB)

(b) Converet SDP into LMI form SDP with small mat. variables — SparsePOP(MATLAB)

(c) Solve LMI form SDP by the primal-dual interior-point method — SeDuMi(MATLAB)

- SDP could become large-scale even when POP is small (say $n = 20$, $m = 20$).
- Sparsity is exploited in (a) too.
- Both lower and upper bounds for the optimal value are obtained.

A POP alkyl from globalib

$$\begin{aligned}
\text{min} \quad & -6.3x_5x_8 + 5.04x_2 + 0.35x_3 + x_4 + 3.36x_6 \\
\text{sub.to} \quad & -0.820x_2 + x_5 - 0.820x_6 = 0,
\end{aligned}$$

$$0.98x_4 - x_7(0.01x_5x_{10} + x_4) = 0, \; -x_2x_9 + 10x_3 + x_6 = 0,$$

$$x_5x_{12} - x_2(1.12 + 0.132x_9 - 0.0067x_9^2) = 0,$$

$$x_8x_{13} - 0.01x_9(1.098 - 0.038x_9) - 0.325x_7 = 0.574,$$

$$x_{10}x_{14} + 22.2x_{11} = 35.82, \; x_1x_{11} - 3x_8 = -1.33,$$

$$\text{lbd}_i \leq x_i \leq \text{ubd}_i \; (i = 1, 2, \ldots, 14).$$

- 14 variables, 7 poly. equality constraints with deg. 3.

Sparse+Conversion			Dense (Lasserre)		
ϵ_{obj}	ϵ_{feas}	cpu	ϵ_{obj}	ϵ_{feas}	cpu
5.6e-10	2.0e-08	23.0	out of	memory	

ϵ_{obj} = approx.opt.val. – lower bound for opt.val.
ϵ_{feas} = the maximum error in the equality constraints

- Global optimality is guaranteed with high accuracy.

A POP ex2_1_8 from globalib

min nonconvex diag. quad. funct. + linear funct.

sub.to 10 sparse linear equalities

lbd$_i \leq x_i \leq$ ubd$_i$ $(i = 1, 2, \ldots, 24)$.

Sparse+Conversion			Dense (Lasserre)		
ϵ_{obj}	ϵ_{feas}	cpu	ϵ_{obj}	ϵ_{feas}	cpu
5.0e-9	1.3e-11	20.0	5.8e-10	3.0e-12	288.8

$\epsilon_{\text{obj}} =$ approx.opt.val. $-$ lower bound for opt.val.

$\epsilon_{\text{feas}} =$ the maximum error in the equality constraints

- Global optimality is guaranteed with high accuracy.

Contents

SDP O (polynomial SDP): min $f_0(\boldsymbol{x})$ sub.to $\boldsymbol{F}(\boldsymbol{x}) \succeq \boldsymbol{O}$.

$$f_0(\boldsymbol{x}) \ : \ \text{a polynomial in } \boldsymbol{x} \in \mathbb{R}^m$$

$$\boldsymbol{F} \ : \ \mathbb{R}^m \rightarrow \mathcal{S}^n, \ F_{ij}(\boldsymbol{x}) \ : \ \text{a polynomial in } \boldsymbol{x} \in \mathbb{R}^m$$

$\boldsymbol{A}_*$: the sparsity pattern matrix;

$[A_*]_{ij} = 0$ if $F_{ij}(\boldsymbol{x}) \equiv 0$, $[A_*]_{ij} = *$ otherwise

Assumption. $\boldsymbol{A}_*$ allows a sparse Cholesky factorization.

$\Downarrow$

positive semidefinite matrix completion technique

SDP C (poly. SDP with multiple but smaller SDP cones:

$$\min \ f_0(\boldsymbol{x}) \text{ sub.to } \boldsymbol{F}_p(\boldsymbol{x}) + \sum_{k=1}^{\ell} \boldsymbol{B}_{pk} z_k \succeq \boldsymbol{O} \ (p = 1, \ldots, \ell).$$

$$\boldsymbol{F}_p \ : \ \mathbb{R}^m \rightarrow \mathcal{S}^{n_p}, \ \boldsymbol{B}_{pk} \in \mathcal{S}^{n_p}.$$

$n_p << n$ under Assumption.

SDP O (tridiag. quad. SDP): min $\sum_{i=1}^{n} c_i x_i$ sub.to $\boldsymbol{F}(\boldsymbol{x}) \succeq \boldsymbol{O}$.

$$\boldsymbol{F} \; : \; \mathbb{R}^n \rightarrow \mathcal{S}^n,$$ each element F_{ij} is quadratic or linear;

$$F_{ij}(\boldsymbol{x}) = \begin{cases} d_i - x_i^2 & \text{if } i = j, \\ (a_i - 0.5)x_i + (b_i - 0.5)x_{i+1} & \text{if } i \leq n-1, \ j = i+1, \\ (a_j - 0.5)x_j + (b_j - 0.5)x_{j+1} & \text{if } j \leq n-1, \ i = j+1, \\ 0 & \text{otherwise.} \end{cases}$$

All $a_i, \ b_i, \ c_i, \ d_i$ are chosen randomly from $[0, 1]$.

$$\Downarrow$$

the sparsity p. mat. $\boldsymbol{A}_*$ — tridiagonal $\Rightarrow$ sparse Cholesky fact.

SDP C (quad. SDP with multiple but smaller SDP cones):

$$\text{min} \ \sum_{i=1}^{n} c_i x_i \ \text{sub.to} \ \boldsymbol{F}_p(\boldsymbol{x}) + \sum_{k=1}^{n-1} \boldsymbol{B}_{pk} z_k \succeq \boldsymbol{O} \ (p = 1, \ldots, n-1).$$

$$\boldsymbol{F}_p \; : \; \mathbb{R}^m \rightarrow \mathcal{S}^2, \ \boldsymbol{B}_{pk} \in \mathcal{S}^2$$

- We will apply a (linear) SDP relaxation for poly. SDP to SDP O and SDP C, and compare their numerical results.

SDP O (tridiag. quad. SDP): min $\sum_{i=1}^{n} c_i x_i$ sub.to $\boldsymbol{F}(\boldsymbol{x}) \succeq \boldsymbol{O}$.

$\boldsymbol{F} \quad : \quad \mathbb{R}^n \to \mathcal{S}^n$, each element F_{ij} is quadratic or linear;

$$F_{ij}(\boldsymbol{x}) = \begin{cases} d_i - x_i^2 & \text{if } i = j, \\ (a_i - 0.5)x_i + (b_i - 0.5)x_{i+1} & \text{if } i \leq n-1,\ j = i+1, \\ (a_j - 0.5)x_j + (b_j - 0.5)x_{j+1} & \text{if } j \leq n-1,\ i = j+1, \\ 0 & \text{otherwise.} \end{cases}$$

All $a_i,\ b_i,\ c_i,\ d_i$ are chosen randomly from $[0, 1]$.

	SDP O, no conversion		SDP C, conversion	
n	sizeA	cpu	sizeA	cpu
50	1325×5101	28.74	197×637	0.36
100	5150×20201	2874.45	397×1287	0.62
200			797×2587	1.38
400			1597×5187	2.70
800			3197×10387	6.29

Contents

1. Conversion of a large scale SDP into an SDP having small matrix variables
2. Two different methods:
   - Conversion to Equality form SDP
   - Conversion to LMI form SDP
3. Some applications to SDP relaxation and successful numerical results
4. In general, it is often difficult to solve SDPs arising from SDP relaxation of POPs and polynomial SDPs; too large to solve, numerical difficulty.

Thank you!