Conversion Methods for Large Scale SDPs to Exploit Their Structured Sparsity

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Equality standard form SDP:

min $\boldsymbol{A}_0 \bullet \boldsymbol{X}$ sub.to $\boldsymbol{A}_p \bullet \boldsymbol{X} = b_p \ (p = 1, \ldots, m), \ \mathcal{S}^n \ni \boldsymbol{X} \succeq \boldsymbol{O}$

Here

$\boldsymbol{A}_p \in \mathcal{S}^n$ the linear space of $n \times n$ symmetric matrices

with the inner product $\boldsymbol{A}_p \bullet \boldsymbol{X} = \sum_{i,\ j} [A_p]_{ij} X_{ij}$.

$b_p \in \mathbb{R}, \ \boldsymbol{X} \succeq \boldsymbol{O} \ \Leftrightarrow \ \boldsymbol{X} \in \mathcal{S}^n$ is positive semidefinite.

Lots of Applications to Various Problems

- Systems and control theory — Linear Matrix Inequality
- SDP relaxations of combinatorial and nonconvex problems
  - Max cut and max clique problems
  - Quadratic assignment problems
  - Polynomial optimization problems
- Robust optimization
- Quantum chemistry
- Moment problems (applied probability)
- Sensor network localization problem — later
- . . .

Equality standard form SDP:

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SDP can be large-scale easily

- $n \times n$ mat. variable $\boldsymbol{X}$ involves $n(n+1)/2$ real variables; $n = 2000 \Rightarrow n(n+1)/2 \approx 2$ million
- m linear equality constraints or m $\boldsymbol{A}_p$'s $\in \mathcal{S}^n$

◊ How can we solve a larger scale SDP?

(a) Use more powerful computer system such as clusters and grids of computers — parallel computation.
(b) Develop new numerical methods for SDPs.
(c) Improve primal-dual interior-point methods.
(d) Convert a large sparse SDP to an SDP which existing pdipms can solve efficiently:
- multiple but small size mat. variables.
- a sparse Schur complement mat. (a coef. mat. of a sys. of equations solved at ∀ iteration of the pdipm).

Outline of the conversion

sparsity used	A large scale and structured sparse SDP	technique
aggregated sparsity	$\Downarrow$	positive definite mat. completion
	An SDP with small SDP cones and shared variables among SDP cones	
correlative sparsity	$\Downarrow$ $\Downarrow$	conversion to LMI form SDP or conversion to Equality form SDP
	A c-sparse SDP with small matrix variables (*i.e.*, small SDP cones)	

An SDP example — Conversion makes a critical difference

min	$\sum_{p=1}^{m} x_p + \boldsymbol{I} \bullet \boldsymbol{X}$
sub.to	$a_p x_p + \boldsymbol{A}_p \bullet \boldsymbol{X} = 2, x_p \geq 0 \ (p = 1, \ldots, m), \ \boldsymbol{X} \succeq \boldsymbol{O}$.

Here $a_p \in (0, 1)$ and $\boldsymbol{A}_p \in \mathcal{S}^k$ are generated randomly.

		SeDuMi	conv.+SeDuMi
m	k	cpu time in sec.	cpu time in sec.
1000	10	29.6	4.3
2000	10	360.4	10.3
4000	10		20.9

SeDuMi — one of the most popular software for SDPs

- Low rank update? But the rank of dense column $= 10(10 + 1)/2 = 55$.
- Application to sensor network localization — later

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Equality standard form SDP:

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$\boldsymbol{A}_*$: $n \times n$ aggregated sparsity pattern mat.

$$[A_*]_{ij} = \begin{cases} \star & \text{if } i = j \text{ or } [A_p]_{ij} \neq 0 \text{ for some } p = 0, \ldots, m, \\ 0 & \text{otherwise} \end{cases}$$

SDP : a-sparse if $\boldsymbol{A}_*$ allows a sparse Cholesky factorization

Two typical cases

1: bandwidth along diagonal 2 : arrow $\searrow$

$$\boldsymbol{A}_* = \begin{pmatrix} \star & \star & 0 & 0 & 0 \\ \star & \star & \star & 0 & 0 \\ 0 & \star & \star & \star & 0 \\ 0 & 0 & \star & \star & \star \\ 0 & 0 & 0 & \star & \star \end{pmatrix} \qquad \boldsymbol{A}_* = \begin{pmatrix} \star & 0 & 0 & 0 & \star \\ 0 & \star & 0 & 0 & \star \\ 0 & 0 & \star & 0 & \star \\ 0 & 0 & 0 & \star & \star \\ \star & \star & \star & \star & \star \end{pmatrix}$$

- $\boldsymbol{X}$: fully dense, so standard pdipms can not effectively utilize this type of sparsity $\Rightarrow$ pos.def.mat.completion

Equality standard form SDP:

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SDP : a-sparse if $\boldsymbol{A}_*$ allows a sparse Cholesky factorization

$\Downarrow$

$G(N, E)$: the asp graph, an undirected graph with

$N = \{1, \ldots, n\}, \ E = \{(i, j) : [A_*]_{ij} = \star$ and $i < j\}$.

$G(N, \overline{E})$: a chordal extension of $G(N, E)$.

$C_1, \ldots, C_\ell \subset N$: the family of maximal cliques of $G(N, \overline{E})$.

SDP $\equiv$ an SDP with shared variables among small SDP cones:

min	$\sum_{(i,j) \in \widetilde{E}} [A_0]_{ij} X_{ij}$
sub.to	$\sum_{(i,j) \in \widetilde{E}} [A_p]_{ij} X_{ij} = b_p \ (\forall p), \ \boldsymbol{X}(C_r) \succeq \boldsymbol{O} \ (r = 1, \ldots, \ell)$,

where $\boldsymbol{X}(C_r)$: the submatrix of $\boldsymbol{X}$ consisting of $X_{ij} \ (i, j \in C_r)$.

Here $\widetilde{E} = \{(i, j) : (i, j), (j, i) \in \overline{E}$ or $i = j\} \Longrightarrow$ Section 3.

Equality standard form SDP:

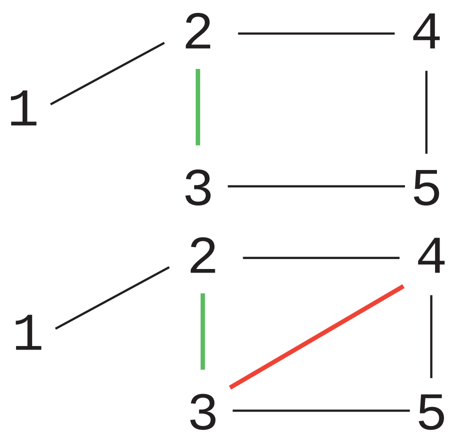
min $\boldsymbol{A}_0 \bullet \boldsymbol{X}$ sub.to $\boldsymbol{A}_p \bullet \boldsymbol{X} = b_p \; (p = 1, \ldots, m), \; \mathcal{S}^n \ni \boldsymbol{X} \succeq \boldsymbol{O}$

$\boldsymbol{A}_*$: $n \times n$ aggregated sparsity pattern mat.

$$[A_*]_{ij} = \begin{cases} \star & \text{if } i = j \text{ or } [A_p]_{ij} \neq 0 \text{ for some } p = 0, \ldots, m, \\ 0 & \text{otherwise} \end{cases}$$

SDP : a-sparse if $\boldsymbol{A}_*$ allows a sparse Cholesky factorization

$$\boldsymbol{A}_* = \begin{pmatrix} \star & \star & 0 & 0 & 0 \\ \star & \star & \star & \star & 0 \\ 0 & \star & \star & 0 & \star \\ 0 & \star & 0 & \star & \star \\ 0 & 0 & \star & \star & \star \end{pmatrix}$$



$G(N, E)$

$\Downarrow$

$G(N, \overline{E})$ chordal

max. cliques

$\{1, 2\}$,$\{2, 3, 4\}$,$\{3, 4, 5\}$

$\widetilde{E} = \{\star$'s & 0's $\}$

min $\sum_{(i,j) \in \widetilde{E}} [A_0]_{ij} X_{ij}$ sub.to $\sum_{(i,j) \in \widetilde{E}} [A_p]_{ij} X_{ij} = b_p,$

$$\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}, \begin{pmatrix} X_{22} & X_{23} & X_{24} \\ X_{32} & X_{33} & X_{34} \\ X_{42} & X_{43} & X_{44} \end{pmatrix}, \begin{pmatrix} X_{33} & X_{34} & X_{35} \\ X_{43} & X_{44} & X_{45} \\ X_{53} & X_{54} & X_{55} \end{pmatrix} \succeq \boldsymbol{O}$$

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SDP with small matrix variables:

$$\min \quad \sum_{r=1}^{\ell} \boldsymbol{A}_{0r} \bullet \boldsymbol{X}_r$$
$$\text{sub.to} \quad \sum_{r=1}^{\ell} \boldsymbol{A}_{pr} \bullet \boldsymbol{X}_r = b_p \ (p = 1, \ldots, m), \ \boldsymbol{X}_r \succeq \boldsymbol{O} \ (\forall r)$$

$\Downarrow$

$$\boldsymbol{A}_{p\diamond} = \text{diag}\left(\boldsymbol{A}_{p1}, \ldots, \boldsymbol{A}_{p\ell}\right), \ \boldsymbol{X}_\diamond = \text{diag}\left(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_\ell\right),$$
$$\boldsymbol{A}_{p\diamond} \bullet \boldsymbol{X}_\diamond = \sum_{r=1}^{\ell} \boldsymbol{A}_{pr} \bullet \boldsymbol{X}_r.$$

SDP: min $\boldsymbol{A}_{0\diamond} \bullet \boldsymbol{X}_\diamond$ sub.to $\boldsymbol{A}_{p\diamond} \bullet \boldsymbol{X}_\diamond = b_p \ (\forall p), \ \boldsymbol{X}_\diamond \succeq \boldsymbol{O}$

$m \times m$ $\boldsymbol{R}_*$: correlative sparsity pattern (csp) mat.

$$[R_*]_{pq} = \begin{cases} 0 & \text{if } \boldsymbol{A}_{p\diamond} \text{ and } \boldsymbol{A}_{q\diamond} \text{ are bw-comp}, \\ \star & \text{otherwise.} \end{cases}$$

$\boldsymbol{A}_{p\diamond}$ and $\boldsymbol{A}_{q\diamond}$: block-wise complementary

$\Updownarrow$

$\boldsymbol{A}_{pr} = \boldsymbol{O}$ or $\boldsymbol{A}_{qr} = \boldsymbol{O}$ for every $r = 1, \ldots, \ell$;

SDP with small matrix variables:

$$\begin{array}{ll} \text{min} & \sum_{r=1}^{\ell} \boldsymbol{A}_{0r} \bullet \boldsymbol{X}_r \\ \text{sub.to} & \sum_{r=1}^{\ell} \boldsymbol{A}_{pr} \bullet \boldsymbol{X}_r = b_p \ (p = 1, \ldots, m), \ \boldsymbol{X}_r \succeq \boldsymbol{O} \ (\forall r) \end{array}$$

$\Downarrow$

$$\boldsymbol{A}_{p\diamond} = \text{diag}\left(\boldsymbol{A}_{p1}, \ldots, \boldsymbol{A}_{p\ell}\right), \ \boldsymbol{X}_{\diamond} = \text{diag}\left(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_\ell\right),$$

$$\boldsymbol{A}_{p\diamond} \bullet \boldsymbol{X}_{\diamond} = \sum_{r=1}^{\ell} \boldsymbol{A}_{pr} \bullet \boldsymbol{X}_r.$$

SDP: min $\boldsymbol{A}_{0\diamond} \bullet \boldsymbol{X}_{\diamond}$ sub.to $\boldsymbol{A}_{p\diamond} \bullet \boldsymbol{X}_{\diamond} = b_p \ (\forall p), \ \boldsymbol{X}_{\diamond} \succeq \boldsymbol{O}$

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$$\begin{array}{l} \boldsymbol{A}_{1\diamond} = \text{diag}(\boldsymbol{A}_{11}, \ \boldsymbol{O}, \ \boldsymbol{O}, \ \boldsymbol{O}) \\ \boldsymbol{A}_{2\diamond} = \text{diag}(\boldsymbol{O}, \boldsymbol{A}_{22}, \ \boldsymbol{O}, \ \boldsymbol{O}) \\ \boldsymbol{A}_{3\diamond} = \text{diag}(\boldsymbol{O}, \ \boldsymbol{O}, \boldsymbol{A}_{33}, \ \boldsymbol{O}) \\ \boldsymbol{A}_{4\diamond} = \text{diag}(\boldsymbol{A}_{41}, \boldsymbol{A}_{42}, \boldsymbol{A}_{43}, \boldsymbol{A}_{44}) \end{array} \Rightarrow \boldsymbol{R}_* = \begin{pmatrix} \star & 0 & 0 & \star \\ 0 & \star & 0 & \star \\ 0 & 0 & \star & \star \\ \star & \star & \star & \star \end{pmatrix}$$

$\exists$ sparse Cholesky factorization

SDP with small matrix variables:

$$\begin{array}{ll} \text{min} & \sum_{r=1}^{\ell} \boldsymbol{A}_{0r} \bullet \boldsymbol{X}_r \\ \text{sub.to} & \sum_{r=1}^{\ell} \boldsymbol{A}_{pr} \bullet \boldsymbol{X}_r = b_p \ (p = 1, \ldots, m), \ \ \boldsymbol{X}_r \succeq \boldsymbol{O} \ (\forall r) \end{array}$$

$\Downarrow$

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fully dense

SDP with small matrix variables:

$$\min \quad \sum_{r=1}^{\ell} \boldsymbol{A}_{0r} \bullet \boldsymbol{X}_r$$
$$\text{sub.to} \quad \sum_{r=1}^{\ell} \boldsymbol{A}_{pr} \bullet \boldsymbol{X}_r = b_p \ (p = 1, \ldots, m), \ \boldsymbol{X}_r \succeq \boldsymbol{O} \ (\forall r)$$

$\Downarrow$

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- $\boldsymbol{R}_* =$ the sparsity pattern of the Schur complement mat. = a coef. mat. of equations solved at $\forall$ iteration of the pdipm by the Cholesky fact.

SDP : c-sparse if $\boldsymbol{R}_*$ allows a sparse Cholesky factorization

c-sparse SDP with small mat. variables — target for conv.

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Outline of the conversion

sparsity used	A large scale and structured sparse SDP	technique
aggregated sparsity	⇓	positive definite mat. completion
	An SDP with small SDP cones and shared variables among SDP cones	
correlative sparsity	⇓ ⇓	conversion to LMI form SDP or conversion to Equality form SDP
	A c-sparse SDP with small matrix variables (*i.e.,* small SDP cones)	

SDP with shared variables among SDP cones

$$\min \sum_{(i,j)\in\widetilde{E}} [A_0]_{ij}X_{ij} \text{ sub.to } \sum_{(i,j)\in\widetilde{E}} [A_p]_{ij}X_{ij} = b_p \ (p=1,\ldots,m),$$
$$\boldsymbol{X}(C_r) \succeq \boldsymbol{O} \ (r=1,\ldots,\ell),$$

$C_1,\ldots,C_r$: the max. cliques of a chordal graph $G(N,\overline{E})$
$\widetilde{E} = \{(i,j) : (i,j),(j,i) \in \overline{E}$ or $i=j\}$.

3-1. Conversion to a c-sparse LMI form SDP

Represent each $\boldsymbol{X}(C_r)$ as

$$\boldsymbol{X}(C_r) = \sum_{i,j\in C_r, i\le j} \boldsymbol{E}^{ij}(C_r)X_{ij},$$

where $\boldsymbol{E}^{ij}(C_r)$: a sym. mat. with 1 at the (i,j)th, (j,i)th elements and 0 elsewhere. Then, a c-sparse LMI form SDP having eq. const.

$$\min \sum_{(i,j)\in\widetilde{E}} [A_0]_{ij}X_{ij} \text{ sub.to } \sum_{(i,j)\in\widetilde{E}} [A_p]_{ij}X_{ij} = b_p \ (\forall p),$$
$$\textstyle\sum_{i,j\in C_r, i\le j} \boldsymbol{E}^{ij}(C_r)X_{ij} \succeq \boldsymbol{O} \ (\forall r).$$

SDP with shared variables among SDP cones

$$\min \sum_{(i,j)\in\widetilde{E}} [A_0]_{ij} X_{ij} \text{ sub.to } \sum_{(i,j)\in\widetilde{E}} [A_p]_{ij} X_{ij} = b_p \ (p = 1,\ldots,m),$$

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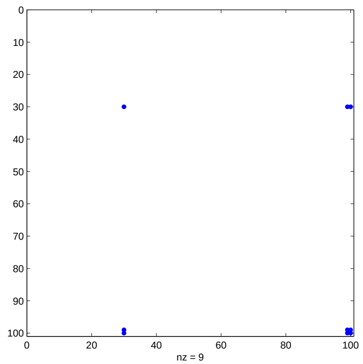
$C_1,\ldots,C_r$: the max. cliques of a chordal graph $G(N,\overline{E})$
$\widetilde{E} = \{(i,j) : (i,j),(j,i) \in \overline{E}$ or $i = j\}$.

3-1. Conversion to a c-sparse LMI form SDP: Example

$n = 100$, $m = 98$, $C_r = \{r, 99, 100\}$ $(1 \le r \le 98)$.

$A_r =$

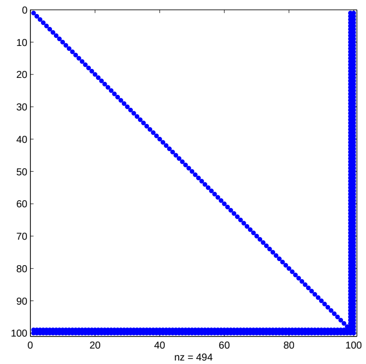


$=$

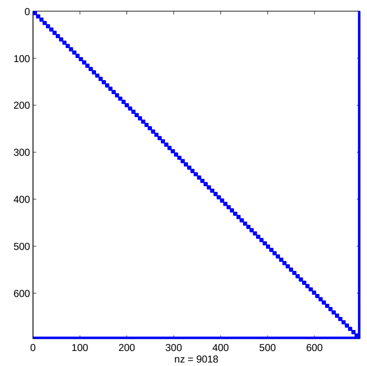
r 99 100
$*$ $*$ $*$

$*$ $*$ $*$
$*$ $*$ $*$

$A_* =$



R_* of LMI form SDP =



SDP with shared variables among SDP cones

$$\min \sum_{(i,j)\in\widetilde{E}} [A_0]_{ij}X_{ij} \text{ sub.to } \sum_{(i,j)\in\widetilde{E}} [A_p]_{ij}X_{ij} = b_p \ (p = 1,\ldots,m),$$
$$\boldsymbol{X}(C_r) \succeq \boldsymbol{O} \ (r = 1,\ldots,\ell),$$

$C_1,\ldots,C_r$: the max. cliques of a chordal graph $G(N,\overline{E})$
$\widetilde{E} = \{(i,j) : (i,j),(j,i) \in \overline{E} \text{ or } i = j\}$.

3-2. Conversion to a c-sparse equality form SDP

We can rewite SDP as

Equality form SDP with indep. mat. var. $\widetilde{\boldsymbol{X}}_r \ (r = 1,\ldots,\ell)$

$$\begin{array}{ll} \min & \sum_{r=1}^{\ell} \widetilde{\boldsymbol{A}}_{0r} \bullet \widetilde{\boldsymbol{X}}_r \\ \text{sub.to} & \sum_{r=1}^{\ell} \widetilde{\boldsymbol{A}}_{pr} \bullet \widetilde{\boldsymbol{X}}_r = b_p \ (p = 1,\ldots,m), \end{array}$$

equalities to identify $\exists$ elements of $\widetilde{\boldsymbol{X}}_r \ (r = 1,\ldots,\ell)$,

$$\widetilde{\boldsymbol{X}}_r \succeq \boldsymbol{O} \ (r = 1,\ldots,\ell).$$

- Various choices for $\widetilde{\boldsymbol{A}}_{pr}$ and equalities.
- How do we choose them for better c-sparsity?

Various choices for equalities

SDP with shared variables $\Rightarrow \boldsymbol{X}(C_r) \succeq \boldsymbol{O}$ $(r = 1, \ldots, \ell)$,
where $C_1, \ldots, C_r$: the max. cliques of $G(N, \overline{E})$

Equality form SDP $\Rightarrow$ $\widetilde{\boldsymbol{X}}_r \succeq \boldsymbol{O}$ $(r = 1, \ldots, \ell)$
and equalities to identify $\exists$ elements of $\widetilde{\boldsymbol{X}}_r$ $(r = 1, \ldots, \ell)$

Example: $n = 100$, $m = 98$, $C_r = \{r, 99, 100\}$ $(1 \leq r \leq 98)$.

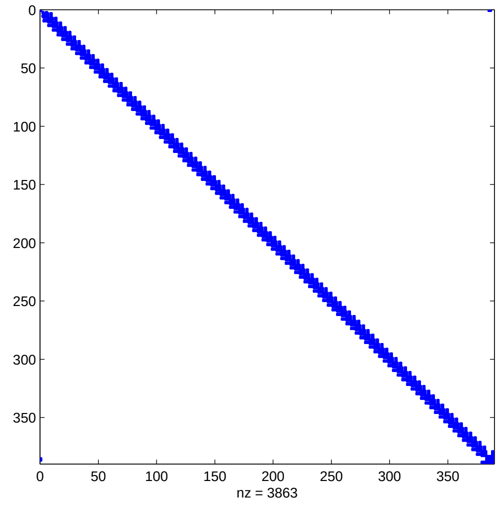
$$\text{each } \widetilde{\boldsymbol{X}}_r = \begin{pmatrix} \star & \star & \star \\ \star & \star & \star \\ \star & \star & \star \end{pmatrix} : 3 \times 3$$

$[\widetilde{X}_r]_{ij} = [\widetilde{X}_1]_{ij}$
$(2 \leq i, j \leq 3)$
$(r = 2, \ldots, 98)$

$[\widetilde{X}_r]_{ij} = [\widetilde{X}_{r-1}]_{ij}$
$(2 \leq i, j \leq 3)$
$(r = 2, \ldots, 98)$

$\boldsymbol{R}_*$ = 389×389, fully dense

$\boldsymbol{R}^*$ =



Various choices for equalities

SDP with shared variables $\Rightarrow \boldsymbol{X}(C_r) \succeq \boldsymbol{O}$ $(r = 1, \ldots, \ell)$,
where $C_1, \ldots, C_r$: the max. cliques of $G(N, \overline{E})$

Equality form SDP $\Rightarrow$ $\widetilde{\boldsymbol{X}}_r \succeq \boldsymbol{O}$ $(r = 1, \ldots, \ell)$
and equalities to identify $\exists$ elements of $\widetilde{\boldsymbol{X}}_r$ $(r = 1, \ldots, \ell)$

- It is often necessary to reduce the number of equalities by combining some cliques.
- Fujisawa, Fukuda, Kojima, Murota and Nakata 2001, 2003, 2006 proposed conversion to an equality form SDP, but correlative sparsity was not exploited $\Longrightarrow$ further study.
- $\exists$ some cases where conversion to a c-sparse LMI form SDP is better, and $\exists$ some cases where conversion to a c-sparse equality form SDP is better.
- Some method to judge which conversion is better for a given problem needs to be studied.

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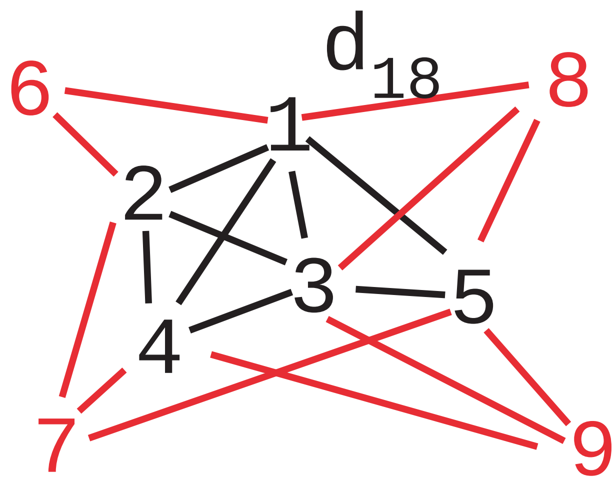
Sensor network localization problem: Let $s = 2$ or 3.

$$
\begin{aligned}
\boldsymbol{x}^p \in \mathbb{R}^s &\quad : \quad \text{unknown location of sensors } (p = 1, 2, \ldots, m), \\
\boldsymbol{x}^r = \boldsymbol{a}^r \in \mathbb{R}^s &\quad : \quad \text{known location of anchors } (r = m+1, \ldots, n), \\
d_{pq} &\quad = \quad \|\boldsymbol{x}^p - \boldsymbol{x}^q\| + \epsilon_{pq} \text{ — given for } (p, q) \in \mathcal{N}, \\
\mathcal{N} &\quad = \quad \{(p, q) : \|\boldsymbol{x}^p - \boldsymbol{x}^q\| \leq \rho = \text{a given radio range}\}
\end{aligned}
$$

Here ϵ_{pq} denotes a noise.

$m = 5,\ n = 9.$
$1, \ldots, 5$: sensors
$6, 7, 8, 9$: anchors



Anchors' positions are fixed.
A distance is given for $\forall$ edge.
Compute locations of sensors.

$\Rightarrow$ Some nonconvex QOPs

- SDP relaxation +? — FSDP by Biswas-Ye '06, ESDP by Wang et al '07, ... for $s = 2$.
- SOCP relaxation — Tseng '07 for $s = 2$.
- ...

Numerical results on 4 methods (a), (b), (c) and (d) applied to a sensor network localization problem with

$m =$ the number of sensors dist. randomly in $[0,1]^2$,
4 anchors located at the corner of $[0,1]^2$,
$\rho =$ radio distance $= 0.1,$ no noise.

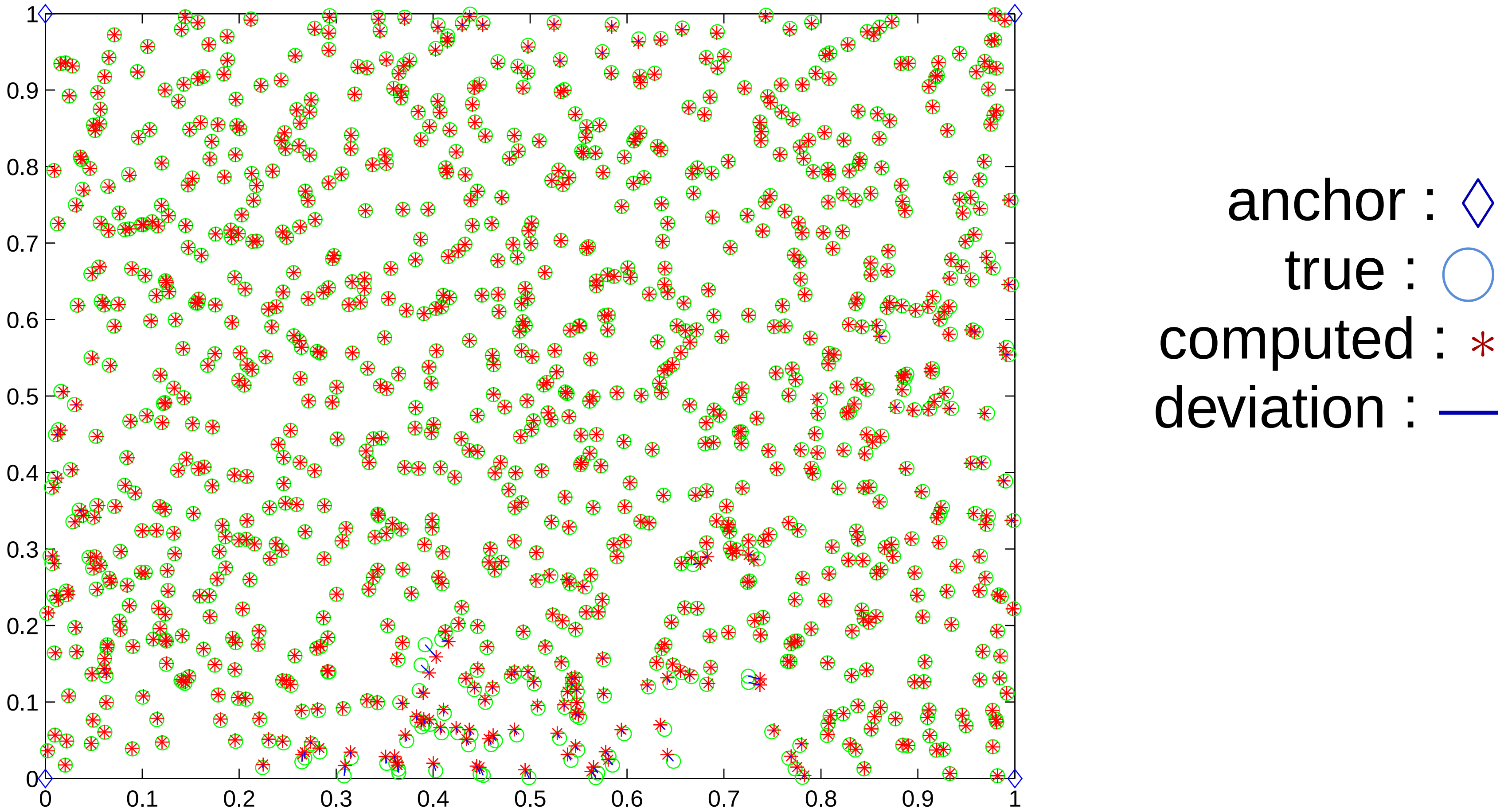
(a) FSDP (b) FSDP + Conv. to LMI form SDP, as strong as (a)
(c) FSDP + Conv. to equality form SDP (Fujisawa-F-K-M-N '01, '03, '06), as strong as (a)
(d) ESDP — a further relaxation of FSDP, weaker than (a);

	SeDuMi cpu time in second			
m	(a)	(b)	(c)	(d)
500	389.1	35.0	405.2	62.5
1000	3345.2	60.4	1317.7	200.3
2000		111.1		1403.9
4000		182.1		11559.8

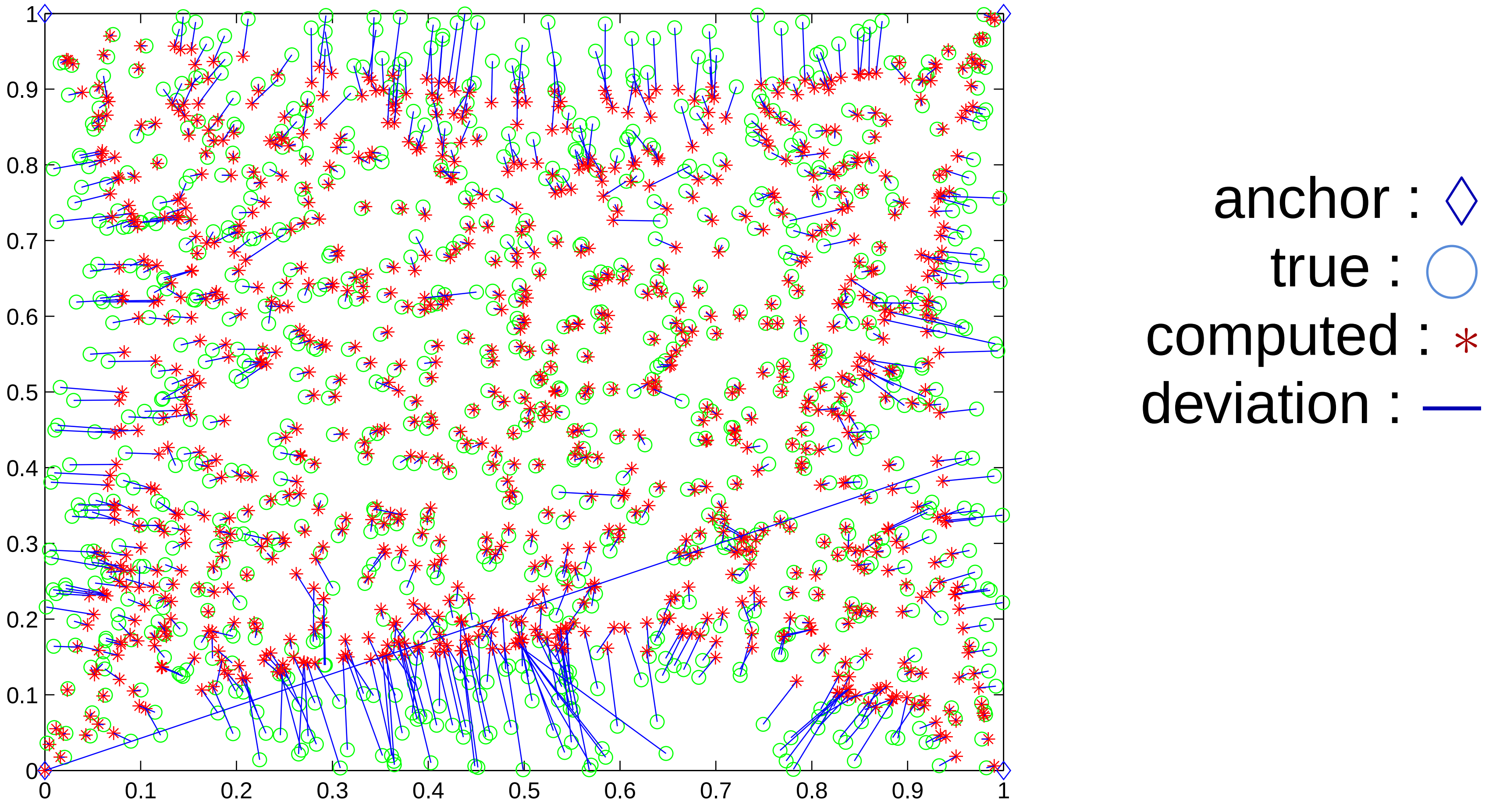
SeDuMi parameters
pars.free=0;
.eps=1.0e-5

$\Rightarrow$ a-sparsity, c-sparsity in (a) and (b)

m = 1000 sensors, (b) FSDP+Conversion to an LMI form SDP

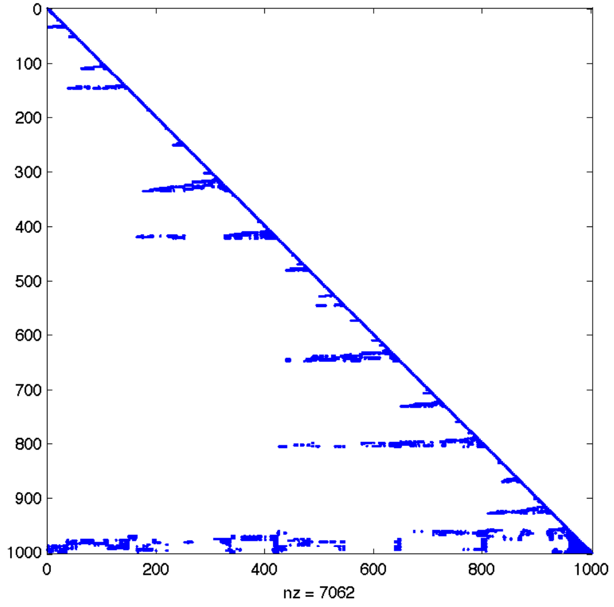


m = 1000 sensors, (d) ESDP



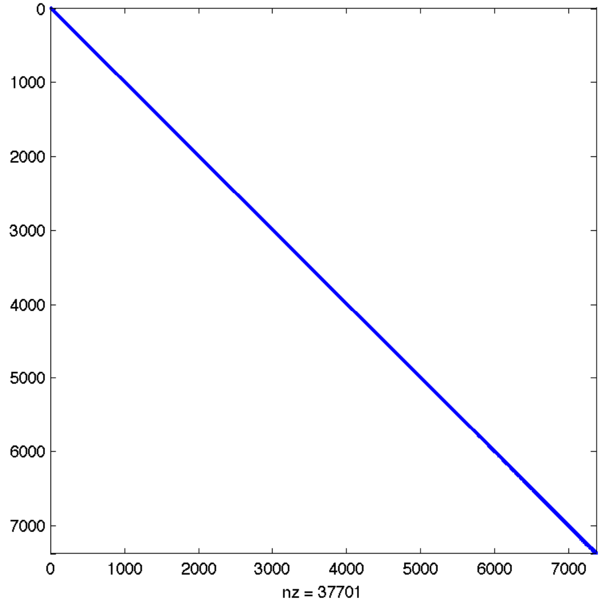
A Cholesky fact. of the a-sparsity pattern matrix $\boldsymbol{A}_*$ with the symm. min. deg. ordering

(a) FSDP (Biswas-Ye '06)



1002×1002, nz = 7062
nz density = 0.014

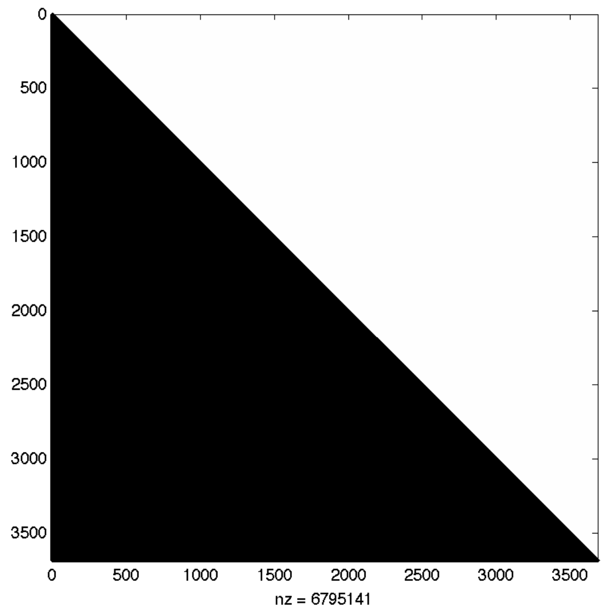
(b) FSDP + Conversion to an LMI form SDP



7381×7381, nz = 37,701
nz density = 0.0014

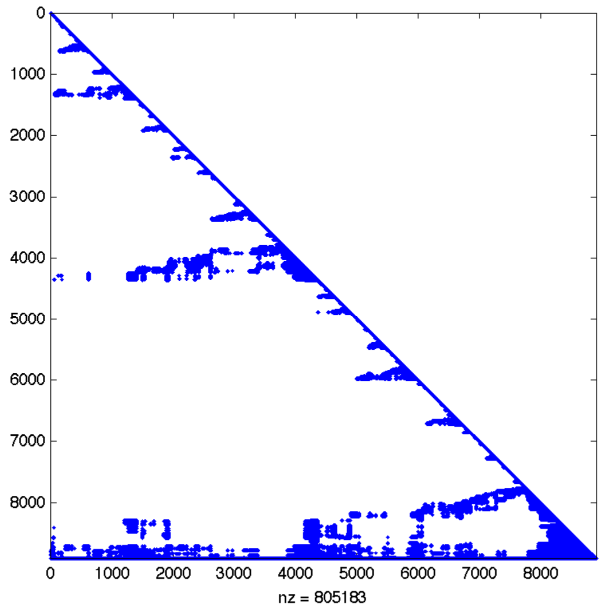
A Cholesky fact. of the c-sparsity pattern matrix $\boldsymbol{R}_*$ (= the Schur comp. matrix) with the symm. min. deg. ordering

(a) FSDP (Biswas-Ye '06)



3686×3686, nz = 6,795,141
nz density = 1.00
3345.2 second

(b) FSDP + Conversion to an LMI form SDP



8916×8916, nz = 805,183
nz density = 0.020
60.4 second

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5. Concluding remarks

1. Large scale SDPs are difficult to solve.
2. Methods which convert a large scale SDP into an SDP having small mat. variables and a sparse Schur complement mat. by exploiting the structured sparsity,
    - aggregated sparsity,
    - correlative sparsity.
3. Two different methods:
    - Conversion to a c-sparse LMI form SDP.
    - Conversion to a c-sparse equality form SDP — further study to exploit correlative sparsity.
4. An application to sensor network localization.