Sparsity in Sum of Squares and SemiDefinite Programmming Relaxations of Polynomial Optimization Problems

Workshop on Polyhedral Computation
Montreal, October 17-20, 2006

Masakazu Kojima
Tokyo Institute of Technology

Contents

Contents

POP: min $f_0(x)$ sub.to $f_j(x) \geq 0$ $(j = 1, \ldots, m)$.

$\mathbb{R}^n$: the n-dim Euclidean space.

$x = (x_1, \ldots, x_n) \in \mathbb{R}^n$: a vector variable.

$f_j(x)$: a multivariate polynomial in $x \in \mathbb{R}^n$ $(j = 0, 1, \ldots, m)$.

Example: $n = 3$

$$
\begin{array}{rl}
\text{min} & f_0(x) \equiv x_1^3 - 2x_1x_2^2 + x_1^2x_2x_3 - 4x_3^2 \\
\text{sub.to} & f_1(x) \equiv -x_1^2 + 5x_2x_3 + 1 \geq 0, \\
& f_2(x) \equiv x_1^2 - 3x_1x_2x_3 + 2x_3 + 2 \geq 0, \\
& f_3(x) \equiv -x_1^2 - x_2^2 - x_3^2 + 1 \geq 0,
\end{array}
$$

$x_1(x_1 - 1) = 0$ (0-1 integer),

$x_2 \geq 0, \ x_3 \geq 0, \ x_2x_3 = 0$ (complementarity).

POP: min $f_0(x)$ sub.to $f_j(x) \geq 0$ $(j = 1, \ldots, m)$.

[1] J.B.Lasserre, "Global optimization with polynomials and the problems of moments", *SIAM J. on Optim.* (2001).

[2] P.A.Parrilo, "Semidefinite programming relaxations for semialgebraic problems", *Math. Prog.* (2003).

[3] D.Henrion and J.B.Lasserre, GloptiPoly.

[4] S.Prajna, A.Parachristodoulou and P.A.Parrilo, SOSTOOLS.

- [1] $\Longrightarrow$ SDP relaxation — primal approach.
- [2] $\Longrightarrow$ SOS relaxation $\Longrightarrow$ SDP — dual approach.

(a) Lower bounds for the optimal value.

(b) Convergence to global optimal solutions in theory.

(c) Expensive to solve large scale POPs in practice.

$\Downarrow$

Exploiting sparsity and parallel computing

POP: $\min f_0(x)$ sub.to $f_j(x) \geq 0$ $(j = 1, \ldots, m)$.

[1] J.B.Lasserre, "Global optimization with polynomials and the problems of moments", *SIAM J. on Optim.* (2001).

[2] P.A.Parrilo, "Semidefinite programming relaxations for semialgebraic problems", *Math. Prog.* (2003).

[3] D.Henrion and J.B.Lasserre, GloptiPoly.

[4] S.Prajna, A.Parachristodoulou and P.A.Parrilo, SOSTOOLS.

Exploiting sparsity to solve larger scale problem in practice

⊙[5] **M. Kojima, S. Kim and H. Waki, "Sparsity in SOS Polynomials",** *Math. Prog.* **(2005)** ⇒ Section 2.

⊙[6] **H. Waki, S. Kim, M. Kojima and M. Muramatsu, "SOS and SDP Relaxations for POPs with Structured Sparsity",** *SIAM J. on Optim* **(2006)** ⇒ Sections 3 and 4.

[7] **H. Waki, S. Kim, M. Kojima and M. Muramatsu,** Sparse-POP **(2005).**

POP: min $f_0(x)$ sub.to $f_j(x) \geq 0$ $(j = 1, \ldots, m)$.

[1] J.B.Lasserre, "Global optimization with polynomials and the problems of moments", *SIAM J. on Optim.* (2001).

[2] P.A.Parrilo, "Semidefinite programming relaxations for semialgebraic problems", *Math. Prog.* (2003).

[3] D.Henrion and J.B.Lasserre, GloptiPoly.

[4] S.Prajna, A.Parachristodoulou and P.A.Parrilo, SOSTOOLS.

Extension to polynomial SDP and SOCP

[8] M. Kojima, "SOS relaxations of polynomial SDPs" (2003).

[9] C.W. Hol and C.W. Hol,"SOS relaxations of polynomial SDPs" (2004).

[10] D. Henrion and J. B. Lasserre, "Convergent relaxations of polynomial matrix inequalities and static output feedback", *IEEE Transactions on Automatic Conrol* (2006).

[11] M. Kojima and M. Muramatsu, "An Extension of SOS Relaxations to POPs over Symmetric Cones ", to applear in *Math. Prog.*

Contents

Notation and symbols

$\mathbb{R}^n$: the n-dim Euclidean space.

$\mathbb{Z}^n_+$: the set of nonnegative n-dim integer vectors.

$x = (x_1, \ldots, x_n) \in \mathbb{R}^n$: a variable vector.

$$f(x) : \text{a polynomial in } x$$

$$\Updownarrow$$

$$\exists \text{ a finite } \mathcal{F} \subset \mathbb{Z}^n_+,\ 0 \neq c(\alpha) \in \mathbb{R}\ (\alpha \in \mathcal{F});$$

$$f(x) = \sum_{\alpha \in \mathcal{F}} c(\alpha) x^{\alpha},$$

where

$$x^{\alpha} = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$$

for $\forall x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\forall \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_+$.

$x^{(0,0)} = 1$ for $\forall x$

Example: $f(x_1, x_2) = -4x_1^3 x_2^4 + 2x_1^4 x_2^3 + 5$

$= -4x^{(3,4)} + 2x^{(4,3)} + 5x^{(0,0)}$,

where $\mathcal{F} = \{(3,4),\ (4,3),\ (0,0)\}$,

$c(3,4) = -4,\ c(4,3) = 2,\ c(0,0) = 5.$

$f(x)$: a nonnegative polynomial $\Leftrightarrow$ $f(x) \geq 0$ $(\forall x \in \mathbb{R}^n)$.
$\mathcal{N}$: the set of nonnegative polynomials in $x \in \mathbb{R}^n$.

$f(x) = \sum_{\alpha \in \mathcal{F}} c(\alpha) x^{\alpha}$: an SOS (Sum of Squares) polynomial

$$\Updownarrow$$

$\exists$ polynomials $g_1(x), \ldots, g_k(x)$; $f(x) = \sum_{i=1}^{k} g_i(x)^2$.

SOS_* : the set of SOS. Obviously, $\mathrm{SOS}_* \subset \mathcal{N}$.

$n = 2$. $f(x_1, x_2) = (x_1^2 - 2x_2 + 1)^2 + (3x_1x_2 + x_2 - 4)^2 \in \mathrm{SOS}_*$.

- In theory, $\mathrm{SOS}_* \subset \mathcal{N}$. $\mathrm{SOS}_* \neq \mathcal{N}$ in general.
- In practice, $f(x) \in \mathcal{N} \backslash \mathrm{SOS}_*$ is rare.
- We replace $\mathcal{N}$ by SOS_* $\Longrightarrow$ SOS Relaxations in Optimization.

$$f(x) = \sum_{\alpha \in \mathcal{F}} c(\alpha) x^{\alpha} \;:\; \text{an SOS (Sum of Squares) polynomial}$$

$$\Updownarrow$$

$$\exists \text{ polynomials } g_1(x), \ldots, g_k(x);\; f(x) = \sum_{i=1}^{k} g_i(x)^2.$$

$$\Updownarrow$$

$$\exists \mathcal{G} \subset \mathbb{Z}^n_+, \exists V \succeq O;\; f(x) \equiv \sum_{\alpha \in \mathcal{G}} \sum_{\beta \in \mathcal{G}} x^{\alpha} V_{\alpha\beta} x^{\beta} \text{ --- } (1)$$

If we fix $\mathcal{G}$, we can compute $V \succeq O$ by solving an LMI (SDP).

Find $V \succeq O$ satisfying $\equiv$ for $\forall x$ $\Rightarrow$

Compare the coefficients of $\forall$ monomial on both side of $\equiv$

$x^{(0,0)} = 1$ for $\forall x$

$$f(x_1, x_2) = 2 - 4x_1^3x_2^4 + 2x_1^4x_2^3 + 5x_1^6x_2^8 - 2x_1^7x_2^7 + 2x_1^8x_2^6$$

$$\equiv \begin{pmatrix} x^{(0,0)} & x^{(3,4)} & x^{(4,3)} \end{pmatrix} \begin{pmatrix} V_{(0,0)(0,0)} & V_{(0,0)(3,4)} & V_{(0,0)(4,3)} \\ V_{(3,4)(0,0)} & V_{(3,4)(3,4)} & V_{(3,4)(4,3)} \\ V_{(4,3)(0,0)} & V_{(4,3)(3,4)} & V_{(4,3)(4,3)} \end{pmatrix} \begin{pmatrix} x^{(0,0)} \\ x^{(3,4)} \\ x^{(4,3)} \end{pmatrix}$$

Here $\mathcal{G} = \{(0,0), (3,4), (4,3)\}$ and $V : 3 \times 3$.

$2 = V_{(0,0)(0,0)}$, $-4 = V_{(0,0)(3,4)} + V_{(3,4)(0,0)}$, $2 = V_{(0,0)(4,3)} + V_{(4,3)(0,0)}$,
$5 = V_{22}$, $-2 = V_{(3,4)(4,3)} + V_{(4,3)(3,4)}$, $2 = V_{(4,3)(4,3)}$.

$f(x) = \sum_{\alpha \in \mathcal{F}} c(\alpha) x^{\alpha}$: an SOS (Sum of Squares) polynomial

$$\Updownarrow$$

$\exists \mathcal{G} \subset \mathbb{Z}^n_+, \exists V \succeq O; \; f(x) = \sum_{\alpha \in \mathcal{G}} \sum_{\beta \in \mathcal{G}} x^{\alpha} V_{\alpha\beta} x^{\beta}$ — (1)

If we fix $\mathcal{G}$, we can check and solve (1) in V by an LMI (SDP).

- How do we choose $\mathcal{G}$ satisfying (1)?
- How do we choose a small size $\mathcal{G}$ satisfying (1) to derive a small size LMI?

$f(x) = \sum_{\alpha \in \mathcal{F}} c(\alpha) x^{\alpha}$: an SOS (Sum of Squares) polynomial

$$\Updownarrow$$

$\exists \mathcal{G} \subset \mathbb{Z}^n_+, \exists V \succeq O;\ f(x) = \sum_{\alpha \in \mathcal{G}} \sum_{\beta \in \mathcal{G}} x^{\alpha} V_{\alpha\beta} x^{\beta}$ — (1)

$\mathcal{F}^{\mathrm{e}} \equiv \{\alpha \in \mathcal{F} : \alpha_i \text{ is even}, \forall i\}$ and $\dfrac{\mathcal{F}^{\mathrm{e}}}{2} \equiv \{\dfrac{\alpha}{2} : \alpha \in \mathcal{F}^{\mathrm{e}}\}$. Then

(1) $\Rightarrow$ $\mathcal{G} \subseteq \mathcal{G}^0 \equiv \left(\text{the convex hull of } \mathcal{F}^{\mathrm{e}}/2\right) \cap \mathbb{Z}^n_+$ (Reznick '78)

- How do we compute $\mathcal{G}^0$?
- How do we eliminate redundant elements from $\mathcal{G}^0$? — later

Example: $f(x_1, x_2) = 2 - 4x_1^3 x_2^4 + 2x_1^4 x_2^3 + 5x_1^6 x_2^8 - 2x_1^7 x_2^7 + 2x_1^8 x_2^6$.

$\mathcal{F}^{\mathrm{e}} = \{(0,0), (6,8), (8,6)\}$, $\mathcal{F}^{\mathrm{e}}/2 = \{(0,0), (3,4), (4,3)\}$,

$\mathcal{G}^0 = \{(0,0), (1,1), (2,2), (3,3), (3,4), (4,3)\}$.

$(1,1), (2,2), (3,3)$: redundant, and can be eliminated. — later

$f(x) = \sum_{\alpha \in \mathcal{F}} c(\alpha) x^{\alpha}$: an SOS (Sum of Squares) polynomial

$$\Updownarrow$$

$\exists \mathcal{G} \subset \mathbb{Z}^n_+, \exists V \succeq O;\ f(x) = \sum_{\alpha \in \mathcal{G}} \sum_{\beta \in \mathcal{G}} x^{\alpha} V_{\alpha\beta} x^{\beta}$ — (1)

$\mathcal{F}^{\mathrm{e}} \equiv \{\alpha \in \mathcal{F} : \alpha_i \text{ is even}, \forall i\}$ and $\dfrac{\mathcal{F}^{\mathrm{e}}}{2} \equiv \{\dfrac{\alpha}{2} : \alpha \in \mathcal{F}^{\mathrm{e}}\}$. Then

(1) $\Rightarrow$ $\mathcal{G} \subseteq \mathcal{G}^0 \equiv \left(\text{the convex hull of } \mathcal{F}^{\mathrm{e}}/2\right) \cap \mathbb{Z}^n_+$ (Reznick '78)

- How do we compute $\mathcal{G}^0$?

We tried to use the software LattE by De Loera et al. based on Barvinok et al. '99, but not successful because

- LattE requires an ineq. description of an input polytope. We combinedly used cdd by K. Fukuda to obtain facets of $\left(\text{the convex hull of } \mathcal{F}^{\mathrm{e}}/2\right)$.
- But the number of facets of $\left(\text{the convex hull of } \mathcal{F}^{\mathrm{e}}/2\right)$ can increase rapidly (exponentially) as $\#\mathcal{F}^{\mathrm{e}}$ increases.

$$f(x) = \sum_{\alpha \in \mathcal{F}} c(\alpha) x^{\alpha} \; : \; \text{an SOS (Sum of Squares) polynomial}$$
$$\Updownarrow$$
$$\exists \mathcal{G} \subset \mathbb{Z}^n_+, \exists V \succeq O; \; f(x) = \sum_{\alpha \in \mathcal{G}} \sum_{\beta \in \mathcal{G}} x^{\alpha} V_{\alpha\beta} x^{\beta} \; \text{— (1)}$$

$\mathcal{F}^{\mathrm{e}} \equiv \{\alpha \in \mathcal{F} : \alpha_i \text{ is even}, \forall i\}$ and $\dfrac{\mathcal{F}^{\mathrm{e}}}{2} \equiv \{\dfrac{\alpha}{2} : \alpha \in \mathcal{F}^{\mathrm{e}}\}$. Then

(1) $\Rightarrow$ $\mathcal{G} \subseteq \mathcal{G}^0 \equiv \left(\text{the convex hull of } \mathcal{F}^{\mathrm{e}}/2\right) \cap \mathbb{Z}^n_+$ (Reznick '78)

- How do we eliminate redundant elements from $\mathcal{G}^0$?

Theorem (Choi at el. '95) Suppose (1) holds for some $\mathcal{G} \subseteq \mathcal{G}^0$. If

$$\beta \in \mathcal{G}, \mathcal{G} \backslash \{\beta\} \neq \emptyset, \; 2\beta \notin \mathcal{F}^{\mathrm{e}}, 2\beta \notin (\mathcal{G} + \mathcal{G} \backslash \{\beta\}) \; \text{— (2)}$$

Then $V_{\alpha\beta} = V_{\beta\alpha} = 0$ for $\forall \alpha \in \mathcal{G} \Rightarrow \mathcal{G} = \mathcal{G} \backslash \{\beta\}$ satisfies (1).

- Let $\mathcal{G} = \mathcal{G}^0$. Checking (2) repeatedly, we eliminate β till we obtain a $\mathcal{G} = \mathcal{G}^*$ such that (2) holds for no β.
- $\mathcal{G}^*$ does not depend on the choice of β in (2); $\mathcal{G}^*$ is unique.

$$f(x) = \sum_{\alpha \in \mathcal{F}} c(\alpha) x^{\alpha} \;:\; \text{an SOS (Sum of Squares) polynomial}$$

$$\Updownarrow$$

$$\exists \mathcal{G} \subset \mathbb{Z}^n_+, \exists V \succeq O;\; f(x) = \sum_{\alpha \in \mathcal{G}} \sum_{\beta \in \mathcal{G}} x^{\alpha} V_{\alpha\beta} x^{\beta} \text{ --- } (1)$$

$\mathcal{F}^{\mathrm{e}} \equiv \{\alpha \in \mathcal{F} : \alpha_i \text{ is even}, \forall i\}$ and $\dfrac{\mathcal{F}^{\mathrm{e}}}{2} \equiv \{\dfrac{\alpha}{2} : \alpha \in \mathcal{F}^{\mathrm{e}}\}$. Then

(1) $\Rightarrow$ $\mathcal{G} \subseteq \mathcal{G}^0 \equiv \left(\text{the convex hull of } \mathcal{F}^{\mathrm{e}}/2\right) \cap \mathbb{Z}^n_+$ (Reznick '78)

Numerical results

$n = 10$, $\mathcal{F}^{\mathrm{e}} \subset \{\alpha \in \mathbb{Z}^{10}_+ : \alpha \leq (4, 4, \ldots, 4)\}$, randomly chosen.

$\#\mathcal{F}^{\mathrm{e}}$	$\#\mathcal{G}^0$	$\#\mathcal{G}^*$	# of facets of $\mathrm{co}\left(\mathcal{F}^{\mathrm{e}}/2\right)$
21	38	23	2,831
31	135	35	19,741
41	354	45	59,543

- $\mathrm{co}\left(\mathcal{F}^{\mathrm{e}}/2\right)$ increases rapidly.
- $\#\mathcal{G}^*$ gets much smaller than $\#\mathcal{G}^0$.

Contents

$\mathcal{P}$: $\min_{x \in \mathbb{R}^n} \; f(x) = \sum_{\alpha \in \mathcal{F}} c(\alpha) x^{\alpha}$

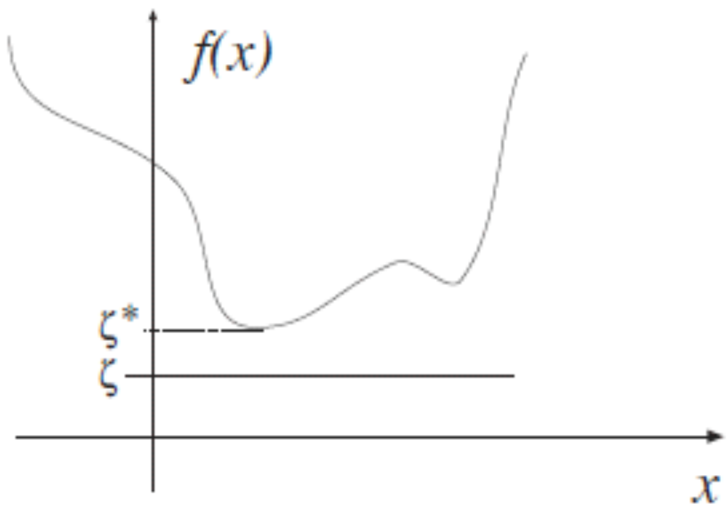
$$\Updownarrow$$

$\mathcal{P}$': max ζ s.t $f(x) - \zeta \geq 0$ $(\forall x \in \mathbb{R}^n)$

$$\Updownarrow$$

$f(x) - \zeta \in \mathcal{N}$ (the nonnegative polynomials)

Here x is a parameter (index) describing inequality constraints.



$\mathcal{P}$: $\min_{x \in \mathbb{R}^n} \; f(x) = \sum_{\alpha \in \mathcal{F}} c(\alpha) x^{\alpha}$

$$\Updownarrow$$

$\mathcal{P}$': max ζ s.t $f(x) - \zeta \geq 0 \; (\forall x \in \mathbb{R}^n)$

$$\Updownarrow$$

$f(x) - \zeta \in \mathcal{N}$ (the nonnegative polynomials)

Here x is a parameter (index) describing inequality constraints.

$\mathrm{SOS}_* \subset \mathcal{N} \;\Downarrow\;$ a subproblem of $\mathcal{P}'$ = a relaxation of $\mathcal{P}$

$\mathcal{P}$": max ζ sub.to $f(x) - \zeta \in \mathrm{SOS}_*$ (SOS polynomials)

- the min.val of $\mathcal{P}$ = the max.val of $\mathcal{P}'$ $\geq$ the max.val of $\mathcal{P}$".
- Use $\mathcal{G}^*$ for $\mathcal{F} \cup \{0\}$ to represent $f(x) - \zeta \in \mathrm{SOS}_*$ as

$$\exists V \succeq O; f(x) - \zeta = \sum_{\alpha \in \mathcal{G}^*} \sum_{\beta \in \mathcal{G}^*} x^{\alpha} V_{\alpha\beta} x^{\beta}$$

as we have discussed in the previous section.

- Then $\mathcal{P}''$ can be solved as an SDP.

- **Exploit the structured sparsity further — next.**

$\mathcal{P}$: $\min_{x \in \mathbb{R}^n} \ f(x) = \sum_{\alpha \in \mathcal{F}} c(\alpha) x^{\alpha}$

H : the sparsity pattern of the Hessian matrix of $f(x)$

$$H_{ij} = \begin{cases} \star \text{ if } i = j \text{ or } \partial^2 f(x)/\partial x_i \partial x_j \not\equiv 0, \\ 0 \text{ otherwise.} \end{cases}$$

$f(x)$: correlatively sparse $\Leftrightarrow$ $\exists$ a sparse Cholesky fact. of H.

(a) A sparse Chol. fact. is characterized as a sparse chordal graph $G(N, E)$; $N = \{1, \ldots, n\}$ and

$$E = \{(i,j) : H_{ij} = \star\} + \text{"fill-in"}.$$

(b) Let $C_1, C_2, \ldots, C_q \subset N$ be the maximal cliques of $G(N, E)$.

Sparse SOS relaxation

max ζ
s.t. $f(x) - \zeta \in \sum_{k=1}^{q}$ (SOS of polynomials in x_i $(i \in C_k)$)

Dense SOS relaxation

max ζ
s.t. $f(x) - \zeta \in$ (SOS of polynomials in x_i $(i \in N)$)

- Sparse relaxation is weaker but less expensive in practice.

Example: Generalized Rosenbrock function.

$$f(x) = \sum_{i=2}^{n} \left(100(x_i - x_{i-1}^2)^2 + (1 - x_i)^2\right).$$

Dense SOS relaxation

max ζ

s.t. $f(x) - \zeta \in$ (SOS of deg-2. poly. in $x_1, x_2, \ldots, x_n$)

• The size of Dense grows very rapidly, so we can't apply Dense to the case $n \geq 20$ in practice.

• The Hessian matrix is sparse (tridiagonal).

• No fill-in in the Cholesky factorization.

• $C_i = \{i-1, i\}$ $(i = 2, \ldots, n)$: the max. cliques.

Sparse SOS relaxation

max ζ

s.t. $f(x) - \zeta \in \sum_{i=2}^{n}$ (SOS of deg-2. poly. in x_{i-1}, x_i)

• The size of Sparse grows linearly in n, and Sparse can process the case $n = 800$ in less than 10 sec.

Contents

POP: $\min\ f_0(x)$ sub.to $f_j(x) \geq 0\ (j = 1, \ldots, m)$.

- Rough sketch of SOS relaxation of POP

"Generalized Lagrangian Dual",
where we take SOS polynomials for Lagrange multipiers.

$+$

"SOS relaxation of unconstrained POPs"

$\Downarrow$

SOS relaxation of POP

- Exploiting sparsity in SOS relaxation of POP

Contents

Numerical results

Software

- SparsePOP (Waki-Kim-Kojima-Muramatsu, 2005)
  — MATLAB program for constructing sparse and dense SDP relaxation problems.
- SeDuMi to solve SDPs.

Hardware

- 2.4GHz Xeon cpu with 6.0GB memory.

G.Rosenbrock function:

$$f(x) = \sum_{i=2}^{n} \left(100(x_i - x_{i-1}^2)^2 + (1 - x_i)^2\right)$$

- Two minimizers on $\mathbb{R}^n$: $x_1 = \pm 1$, $x_i = 1$ $(i \geq 2)$.
- Add $x_1 \geq 0 \Rightarrow$ a single minimizer.

		cpu in sec.	
n	ϵ_{obj}	Sparse	Dense
10	2.5e-08	0.2	10.6
15	6.5e-08	0.2	756.6
200	5.2e-07	2.2	—
400	2.5e-06	3.7	—
800	5.5e-06	6.8	—

$$\epsilon_{\text{obj}} = \frac{|\text{the lower bound for opt. value} - \text{the approx. opt. value}|}{\max\{1, |\text{the lower bound for opt. value}|\}}.$$

An optimal control problem from Coleman et al. 1995

$$\left.\begin{array}{l} \min\ \dfrac{1}{M}\displaystyle\sum_{i=1}^{M-1}(y_i^2 + x_i^2) \\ \text{s.t.}\quad y_{i+1} = y_i + \dfrac{1}{M}(y_i^2 - x_i), \quad (i = 1, \ldots, M-1), \quad y_1 = 1. \end{array}\right\}$$

Numerical results on sparse relaxation

M	# of variables	ϵ_{obj}	ϵ_{feas}	cpu
600	1198	3.4e-08	2.2e-10	3.4
700	1398	2.5e-08	8.1e-10	3.3
800	1598	5.9e-08	1.6e-10	3.8
900	1798	1.4e-07	6.8e-10	4.5
1000	1998	6.3e-08	2.7e-10	5.0

$$\epsilon_{\text{obj}} = \frac{|\text{the lower bound for opt. value} - \text{the approx. opt. value}|}{\max\{1, |\text{the lower bound for opt. value}|\}},$$

ϵ_{feas} = the maximum error in the equality constraints,

cpu : cpu time in sec. to solve an SDP relaxation problem.

alkyl.gms : a benchmark problem from globallib

$$
\begin{array}{ll}
\text{min} & -6.3x_5x_8 + 5.04x_2 + 0.35x_3 + x_4 + 3.36x_6 \\
\text{sub.to} & -0.820x_2 + x_5 - 0.820x_6 = 0, \\
& 0.98x_4 - x_7(0.01x_5x_{10} + x_4) = 0, \\
& -x_2x_9 + 10x_3 + x_6 = 0, \\
& x_5x_{12} - x_2(1.12 + 0.132x_9 - 0.0067x_9^2) = 0, \\
& x_8x_{13} - 0.01x_9(1.098 - 0.038x_9) - 0.325x_7 = 0.574, \\
& x_{10}x_{14} + 22.2x_{11} = 35.82, \\
& x_1x_{11} - 3x_8 = -1.33, \\
& \text{lbd}_i \leq x_i \leq \text{ubd}_i \ (i = 1, 2, \ldots, 14).
\end{array}
$$

		Sparse			Dense (Lasserre)		
problem	n	ϵ_{obj}	ϵ_{feas}	cpu	ϵ_{obj}	ϵ_{feas}	cpu
alkyl	14	5.6e-10	2.0e-08	23.0	out of memory		

$$
\epsilon_{\text{obj}} = \frac{|\text{the lower bound for opt. value} - \text{the approx. opt. value}|}{\max\{1, |\text{the lower bound for opt. value}|\}},
$$

ϵ_{feas} = the maximum error in the equality constraints,

cpu : cpu time in sec. to solve an SDP relaxation problem.

Some other benchmark problems from globallib

		Sparse			Dense (Lasserre)		
problem	n	ϵ_{obj}	ϵ_{feas}	cpu	ϵ_{obj}	ϵ_{feas}	cpu
ex3_1_1	8	6.3e-09	4.7e-04	5.5	0.7e-08	2.5e-03	597.8
st_bpaf1b	10	3.8e-08	2.8e-08	1.0	4.6e-09	7.2e-10	1.7
st_e07	10	0.0e+00	8.1e-05	0.4	0.0e+00	8.8e-06	3.0
st_jcbpaf2	10	1.1e-07	0.0e+00	2.1	1.1e-07	0.0e+00	2.0
ex2_1_3	13	5.1e-09	3.5e-09	0.5	1.6e-09	1.5e-09	7.7
ex9_1_1	13	0.0	4.5e-06	1.5	0.0	9.2e-07	7.7
ex9_2_3	16	0.0e+00	5.7e-06	2.3	0.0e+00	7.5e-06	49.7
ex2_1_8	24	1.0e-05	0.0e+00	304.6	3.4e-06	0.0e+00	1946.6
ex5_2_2_c1	9	1.0e-2	3.2e+01	1.8	1.6e-05	2.1e-01	2.6
ex5_2_2_c2	9	1.0e-02	7.2e+01	2.1	1.3e-04	2.7e-01	3.5

- ex5_2_2_c1 and ex5_2_2_c2 — Dense is better.
- Sparse attains approx. opt. solutions with the same quality as Dense except ex5_2_2_c1 and ex5_2_2_c2.
- Sparse is much faster than Dense in large dim. cases.

Contents

- Lasserre's (dense) relaxation
  — Theoretical convergence but expensive in practice.
- Sparse relaxation (Waki-Kim-Kojima-Muramatsu)
  = Lasserre's (dense) relaxation + sparsity
  — Very powerful in practice
  and theoretical convergence (Lasserre)

- There remain many issues to be studied further.
  - Exploiting sparsity.
  - Large-scale SDPs.
  - Numerical difficulty in solving SDP relaxations of POPs.
  - Polynomial SDPs.

Thank you!

This presentation material is available at

http://www.is.titech.ac.jp/~kojima/talk.html