Sums of Squares Relaxation
of Polynomial Optimization Problems

Dynamical System and Numerical Analysys
In honor of Tien-Yien Li
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- An introduction to the recent development of SOS relaxation for computing global optimal solutions of POPs

Outline
1. POPs (Polynomial Optimization Problems)
2. Nonnegative polynomials and SOS (Sum of Squares) polynomials
3. SOS relaxation of unconstrained POPs
4. Conversion of SOS relaxation into an SDP (Semidefinite Program)
5. Structured sparsity
6. SOS relaxation of constrained POPs
7. Numerical results
8. Concluding remarks

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$\mathbb{R}^n$: the n-dim Euclidean space.

$x = (x_1, \ldots, x_n) \in \mathbb{R}^n$: a vector variable.

$f_j(x)$: a multivariate polynomial in $x \in \mathbb{R}^n$ $(j = 0, 1, \ldots, m)$.

POP: min $f_0(x)$ sub.to $f_j(x) \geq 0$ $(j = 1, \ldots, m)$.

Example: $n = 3$

$$\begin{array}{rl} \text{min} & f_0(x) \equiv x_1^3 - 2x_1x_2^2 + x_1^2x_2x_3 - 4x_3^2 \\ \text{sub.to} & f_1(x) \equiv -x_1^2 + 5x_2x_3 + 1 \geq 0, \\ & f_2(x) \equiv x_1^2 - 3x_1x_2x_3 + 2x_3 + 2 \geq 0, \\ & f_3(x) \equiv -x_1^2 - x_2^2 - x_3^2 + 1 \geq 0. \\ & x_1(x_1 - 1) = 0 \text{ (0-1 integer)}, \\ & x_2 \geq 0,\ x_3 \geq 0,\ x_2x_3 = 0 \text{ (complementarity)}. \end{array}$$

- Various problems can be described as POPs.
- A unified theoretical model for global optimization in nonlinear and combinatorial optimization problems.

Two approaches to SOS and SDP relaxations of POPs

POP: min $f_0(x)$ sub.to $f_i(x) \geq 0$ $(i = 1, \ldots, m)$,

POP $\Rightarrow$ generalized Lagrangian dual

$\Updownarrow$ add valid LMIs — dual — $\Downarrow$

Polynomial SDP — $\Downarrow$ SOS relaxation

$\Downarrow$ linearize (relaxation) — dual — $\Downarrow$

SDP[1] $\Leftrightarrow$ SDP[2]

[1] J.B.Lasserre, "Global optimization with polynomials and the problems of moments", *SIAM J. on Optimization*, 11 (2001) 796–817.

[2] P.A.Parrilo, "Semidefinite programming relaxations for semialgebraic problems". *Math. Prog.*, 96 (2003) 293–320.

Two approaches to SOS and SDP relaxations of POPs

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SDP[1] $\Leftrightarrow$ SDP[2]

(a) Global optimal solutions.

(b) Large-scale SDPs require enormous computation.

(c) Proposed a sparse SDP relaxation
= SDP[1] + "Exploiting structured sparsity".

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$f(x)$: a nonnegative polynomial $\Leftrightarrow$ $f(x) \geq 0$ $(\forall x \in \mathbb{R}^n)$.
$\mathcal{N}$: the set of nonnegative polynomials in $x \in \mathbb{R}^n$.

$f(x)$: an SOS (Sum of Squares) polynomial

$\Updownarrow$

$\exists$ polynomials $g_1(x), \ldots, g_k(x)$; $f(x) = \sum_{i=1}^{k} g_i(x)^2$.

SOS_* : the set of SOS. Obviously, $\mathrm{SOS}_* \subset \mathcal{N}$.
$\mathrm{SOS}_{2r} = \{f \in \mathrm{SOS}_* : \deg f \leq 2r\}$: SOSs with degree ar most $2r$.

$n = 2$. $f(x_1, x_2) = (x_1^2 - 2x_2 + 1)^2 + (3x_1x_2 + x_2 - 4)^2 \in \mathrm{SOS}_4$.

$n = 2$. $f(x_1, x_2) = (x_1x_2 - 1)^2 + x_1^2 \in \mathrm{SOS}_4$.

- In theory, SOS_* (SOS) $\subset \mathcal{N}$. $\mathrm{SOS}_* \neq \mathcal{N}$ in general.
- If $n = 1$, $\mathrm{SOS}_* = \mathcal{N}$. $\{f \in \mathcal{N} : \deg f \leq 2\} \equiv \mathrm{SOS}_2$.
- In practice, $f(x) \in \mathcal{N} \backslash \mathrm{SOS}_*$ is rare.
- So we replace $\mathcal{N}$ by $\mathrm{SOS}_* \Longrightarrow$ SOS Relaxations.

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$\mathcal{P}$: $\min_{x \in \mathbb{R}^n} f(x)$, where f is a polynomial with $\deg f = 2r$

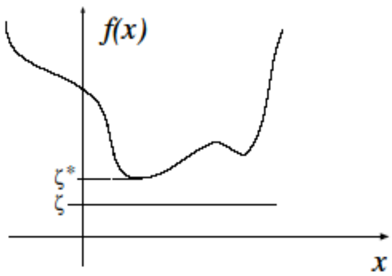
$$\Updownarrow$$

$\mathcal{P}'$: $\max \zeta$ s.t $f(x) - \zeta \geq 0 \; (\forall x \in \mathbb{R}^n)$

$$\Updownarrow$$

$f(x) - \zeta \in \mathcal{N}$ (the nonnegative polynomials)

Here x is an index describing inequality constraints.



$\mathcal{P}$: $\min_{x \in \mathbb{R}^n} f(x)$, where f is a polynomial with deg $f = 2r$

$\Updownarrow$

$\mathcal{P}$': max ζ s.t $f(x) - \zeta \geq 0$ $(\forall x \in \mathbb{R}^n)$

$\Updownarrow$

$f(x) - \zeta \in \mathcal{N}$ (the nonnegative polynomials)

Here x is an index describing inequality constraints.

$\Sigma \subset \text{SOS}_{2r} \subset \text{SOS}_* \subset \mathcal{N}$ $\Downarrow$ a subproblem of $\mathcal{P}'$ = a relaxation of $\mathcal{P}$

$\mathcal{P}$": max ζ sub.to $f(x) - \zeta \in \Sigma$

SOS_* (SOS_{2r} =) the set of SOS polynomials (with degree $\leq 2r$).

- the min.val of $\mathcal{P}$ = the max.val of $\mathcal{P}'$ $\geq$ the max.val of $\mathcal{P}$".
- $\mathcal{P}$" can be solved as an SDP (Semidefinite Program) — next.
- In practice, we can exploit structured sparsity of the Hessian matrix of f to reduce the size of Σ — later.

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What is an SDP (Semidefinite Program)?

- An extension of LP (Linear Program) to the space of symmetric matrices;

variable	a vector $x \Longrightarrow$ a symmetric matrix X.
inequality	$x \geq 0 \Longrightarrow X \succeq O$ (positive semidefinite).

- Can be solved by the interior-point method.
- Lots of applications.

A primal dual pair of LPs:

$$\begin{array}{ll} \text{PLP:} & \max \ a_0 \cdot x \\ & \text{s.t.} \ \ a_p \cdot x = b_p \ (p = 1, \ldots, m), \ x \geq 0. \\ \text{DLP:} & \min \ \sum_{p=1}^m b_p y_p \ \text{ s.t. } \ \sum_{p=1}^m a_p y_p - a_0 \geq 0. \end{array}$$

$a_p \in \mathbb{R}^n \ (p = 0, 1, 2, \ldots, m)$, $b_p \in \mathbb{R} \ (p = 1, 2, \ldots, m)$.
$x \in \mathbb{R}^n$, $y_p \in \mathbb{R} \ (p = 1, 2, \ldots, m)$: variable.
$a_p \cdot x = \sum_{j=1}^n [a_p]_j x_j$ (the inner product).

A primal dual pair of SDPs:

$$\begin{array}{ll} \text{PSDP:} & \max \ A_0 \bullet X \\ & \text{s.t.} \ \ A_p \bullet X = b_p \ (p = 1, \ldots, m), \ X \succeq O. \\ \text{DSDP:} & \min \ \sum_{p=1}^m b_p y_p \ \text{ s.t. } \ \sum_{p=1}^m A_p y_p - A_0 \succeq O. \end{array}$$

$\mathcal{S}^n$: the set of $n \times n$ real symmetric matrices.
$X \succeq O$: $X \in \mathcal{S}^n$ is positive semidefinite.
$A_p \in \mathcal{S}^n \ (p = 0, 1, 2, \ldots, m)$, $b_p \in \mathbb{R} \ (p = 1, 2, \ldots, m)$.
$X \in \mathcal{S}^n$, $y_p \in \mathbb{R} \ (p = 1, 2, \ldots, m)$: variable.
$A_p \bullet X = \sum_{i=1}^n \sum_{j=1}^n [A_p]_{ij} X_{ij}$ (the inner product).

Representation of

$$\mathrm{SOS}_{2r} \equiv \left\{ \sum_{j=1}^{k} g_j(x)^2 : \exists k \geq 1,\ g_j(x) : \text{ degree at most } r \right\} \subset \mathrm{SOS}_*.$$

$\forall$ r-degree poly. $g(x)$ $\exists a \in \mathbb{R}^{d(r)}$; $g(x) = a^T u_r(x)$, where

$$u_r(x) = (1, x_1, x_2, \ldots, x_n, x_1^2, x_1x_2, x_1x_3, \ldots, x_n^2, \ldots, x_1^r, \ldots, x_n^r)^T,$$

(a column vector of a basis of r-degree polynomial),

$$d(r) = \binom{n+r}{r} : \text{ the dimension of } u_r(x).$$

$\Downarrow$

$$\begin{aligned}
\mathrm{SOS}_{2r} &= \left\{ \sum_{j=1}^{k} \left(a_j^T u_r(x)\right)^2 : k \geq 1,\ a_j \in \mathbb{R}^{d(r)} \right\} \\
&= \left\{ u_r(x)^T \left(\sum_{j=1}^{k} a_j a_j^T \right) u_r(x) : k \geq 1,\ a_j \in \mathbb{R}^{d(r)} \right\} \\
&= \left\{ u_r(x)^T V u_r(x) : V \text{ is a positive semidefinite matrix} \right\}.
\end{aligned}$$

Example. $n = 1$, SOS of at most deg.3 polynomials in $x \in \mathbb{R}$.

$$\begin{aligned}
\mathrm{SOS}_6 &\equiv \left\{ \sum_{i=1}^{k} g_i(x)^2 : k \geq 1,\ g_i(x) \text{ is at most deg.3 polynomial} \right\} \\
&= \left\{ \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \end{pmatrix}^T V \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \end{pmatrix} : V \text{ is } 4 \times 4 \text{ psd matrix} \right\}
\end{aligned}$$

Example. $n = 2$, SOS of at most deg.2 polynomials in $x=(x_1, x_2)$.

$$\begin{aligned}
\mathrm{SOS}_4 &\equiv \left\{ \sum_{i=1}^{k} g_i(x)^2 : k \geq 1,\ g_i(x) \text{ is at most deg.2 polynomial} \right\} \\
&= \left\{ \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_1^2 \\ x_1x_2 \\ x_2^2 \end{pmatrix}^T V \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_1^2 \\ x_1x_2 \\ x_2^2 \end{pmatrix} : V \text{ is a } 6 \times 6 \text{ psd matrix} \right\}
\end{aligned}$$

SOS Optimization $\Longrightarrow$ SDP.

Example : $f(x) = -x_1 + 2x_2 + 3x_1^2 - 5x_1^2x_2^2 + 7x_2^4$

$\max \zeta$ sub.to $f(x) - \zeta \in \mathcal{SOS}_4$ (SOS of at most deg. 2 polynomials)

$\Updownarrow$

Sum of Squares

$$\max\ \zeta \quad \text{s.t.} \quad f(x) - \zeta = \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_1^2 \\ x_1x_2 \\ x_2^2 \end{pmatrix}^T \begin{pmatrix} V_{11} & V_{12} & V_{13} & V_{14} & V_{15} & V_{16} \\ V_{12} & V_{22} & V_{23} & V_{24} & V_{25} & V_{26} \\ V_{13} & V_{23} & V_{33} & V_{34} & V_{35} & V_{36} \\ V_{14} & V_{24} & V_{34} & V_{44} & V_{45} & V_{46} \\ V_{15} & V_{25} & V_{35} & V_{45} & V_{55} & V_{56} \\ V_{16} & V_{26} & V_{36} & V_{46} & V_{56} & V_{66} \end{pmatrix} \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_1^2 \\ x_1x_2 \\ x_2^2 \end{pmatrix}$$

$(\forall (x_1, x_2)^T \in \mathbb{R}^n)$, $\quad 6 \times 6\ V \succeq O$

$\Updownarrow$ Compare the coef. of $1,\ x_1,\ x_2,\ x_1^2,\ x_1x_2, x_2^2$ on both side of $=$

SDP (Semidefinite Program)

$$\max\ \zeta \text{ s.t. } \quad -\zeta = V_{11},\ -1 = 2V_{12},\ 2 = 2V_{13},\ 3 = 2V_{14} + V_{22},$$
$$-5 = 2V_{46} + V_{55},\ 7 = V_{66},\ \text{all others}\quad 0 = \cdots\ ,\ V \succeq O$$

In general, each equality constraint is a linear equation in ζ and V.

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$\mathcal{P}$: $\min_{x \in \mathbb{R}^n} f(x)$, where f is a polynomial with $\deg f = 2r$

H : the sparsity pattern of the Hessian matrix of $f(x)$

$$H_{ij} = \begin{cases} \star & \text{if } i = j \text{ or } \partial^2 f(x)/\partial x_i \partial x_j \not\equiv 0, \\ 0 & \text{otherwise.} \end{cases}$$

$f(x)$: correlatively sparse $\Leftrightarrow$ $\exists$ a sparse Cholesky fact. of H.

(a) The sparse Cholesky fact. is characterized as a sparse chordal graph $G(N, E)$; $N = \{1, \ldots, n\}$ and $E \subset N \times N$.
(b) Let $C_1, C_2, \ldots, C_q \subset N$ be the maximal cliques of $G(N, E)$.

Sparse relaxation

max ζ
s.t. $f(x) - \zeta \in \sum_{k=1}^{q}$ (SOS of polynomials in x_i $(i \in C_k)$)

Dense relaxation

max ζ
s.t. $f(x) - \zeta \in$ (SOS of polynomials in x_i $(i \in N)$)

- Sparse relaxation is weaker but less expensive in practice.

Generalized Rosenbrock function + Perturbation.

$$f(x) = \sum_{i=2}^{n} (100(x_i - x_{i-1}^2)^2 + (1 - x_i)^2) + \sum_{i=1}^{n} a_i x_i, \; 0 < a_i < 0.1.$$

- The Hessian matrix is sparse (tridiagonal).

Sparse relaxation

max ζ
s.t. $f(x) - \zeta \in \sum_{i=2}^{n}$ (SOS of 2-deg. poly in x_{i-1}, x_i)

Dense relaxation

max ζ
s.t. $f(x) - \zeta \in$ (SOS of 2-deg. poly in $x_1, x_2, \ldots, x_n$)

Generalized Rosenbrock function + Perturbation.

$$f(x) = \sum_{i=2}^{n} (100(x_i - x_{i-1}^2)^2 + (1 - x_i)^2) + \sum_{i=1}^{n} a_i x_i, \ 0 < a_i < 0.1.$$

- The Hessian matrix is sparse (tridiagonal).

Sparse relaxation
max ζ
s.t. $f(x) - \zeta \in \sum_{i=2}^{n}$ (SOS of 2-deg. poly in $x_{i-1}, \ x_i$)

		cpu in sec.	
n	ϵ_{obj}	sparse	Lasserre's dense
10	1.9e-08	0.2	10.6
15	2.1e-08	0.3	756.6
200	6.8e-08	1.9	—
400	4.1e-08	4.0	—
800	2.0e-08	7.5	—

$$\epsilon_{\text{obj}} = \frac{|\text{the lower bound for opt. value} - \text{the approx. opt. value}|}{\max\{1, |\text{the lower bound for opt. value}|\}}.$$

Broyden's tridiagonal function + Perturbation.

$$f(x) = \sum_{i=2}^{n-1} ((3 - 2x_i)x_i - x_{i-1} - 2x_{i+1} + 1)^2 + \sum_{i=1}^{n} a_i x_i, \ 0 < a_i < 0.1.$$

- The Hessian matrix is sparse.

Sparse relaxation
max ζ
s.t. $f(x) - \zeta \in \sum_{i=2}^{n-1}$ (SOS of 2-deg. poly in $x_{i-1}, \ x_i, \ x_{i+1}$)

		cpu in sec.	
n	ϵ_{obj}	sparse	Lasserre's dense
10	1.9e-08	0.2	15.5
15	2.1e-08	0.3	804.5
200	3.2e-08	3.4	—
400	3.0e-08	6.7	—
800	3.0e-08	13.2	—

$$\epsilon_{\text{obj}} = \frac{|\text{the lower bound for opt. value} - \text{the approx. opt. value}|}{\max\{1, |\text{the lower bound for opt. value}|\}}.$$

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- Rough sketch of SOS relaxation of constrained POPs

"(Generalized) Lagrangian Dual"

$+$

"SOS relaxation of unconstrained POPs"

$\Downarrow$

SOS relaxation of constrained POPs

POP: min $f_0(x)$ sub.to $f_j(x) \geq 0$ $(j = 1, \ldots, m)$

Generalized Lagrange function:

$$L(x, \varphi) = f_0(x) - \varphi_1(x) f_1(x) \cdots - \varphi_m(x) f_m(x).$$

where, $\varphi = (\varphi_1, \ldots, \varphi_m) \in \mathrm{SOS}_*^m$, $\varphi_j \in \mathrm{SOS}_*$ (SOS polynomials).

G. Lag. dual: $\max_{\varphi \in \mathrm{SOS}_*^m} \min_{x \in \mathbb{R}^n} L(x, \varphi)$

$\Updownarrow$

G. Lag. dual: $\max\ \zeta$ s.t $L(x, \varphi) - \zeta \geq 0$ $(\forall x \in \mathbb{R}^n)$, $\varphi \in \mathrm{SOS}_*^m$

SOS relaxation $\Downarrow$

$\max\ \zeta$ s.t $L(x, \varphi) - \zeta \in \mathrm{SOS}_*$, $\varphi \in \mathrm{SOS}_*^m$

a finite size $\Downarrow$ $\Xi \subset \{\varphi(x) = (\varphi_1, \ldots, \varphi_m) : \varphi_j \in \mathrm{SOS}_{2r}\}$ for $\exists r$, $\Sigma \subset \mathrm{SOS}_{2s}$ for $\exists s \geq r$

SOS relaxation: $\max\ \zeta$ s.t $L(x, \varphi) - \zeta \in \Sigma$, $\varphi \in \Xi$

- SOS relaxation can be solved as an SDP.
- As r $\uparrow$, a better lower bound for the opt. val. of POP.
- Sparsity of POP to reduce the sizes of Ξ and Σ; take Ξ so that $L(x, \varphi(x))$ $(\varphi \in \Xi)$ becomes correlatively sparse.
- opt.value of G.Lag.dual $\leq$ opt.value of POP in general.
- "=" holds under moderate assumptions.

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Software

- MATLAB for constructing sparse and dense SDP relaxation problems
- SeDuMi to solve SDPs.

Hardware

- 2.4GHz Xeon cpu with 6.0GB memory.

An optimal control problem from Coleman et al. 1995

$$\left.\begin{array}{l} \min \ \dfrac{1}{M}\displaystyle\sum_{i=1}^{M-1}\left(y_i^2 + x_i^2\right) \\ \text{s.t.} \ \ y_{i+1} = y_i + \dfrac{1}{M}(y_i^2 - x_i), \quad (i = 1, \ldots, M-1), \quad y_1 = 1. \end{array}\right\}$$

Numerical results on sparse relaxation

M	# of variables	ϵ_{obj}	ϵ_{feas}	cpu
600	1198	3.4e-08	2.2e-10	3.4
700	1398	2.5e-08	8.1e-10	3.3
800	1598	5.9e-08	1.6e-10	3.8
900	1798	1.4e-07	6.8e-10	4.5
1000	1998	6.3e-08	2.7e-10	5.0

$$\epsilon_{\text{obj}} = \frac{|\text{the lower bound for opt. value} - \text{the approx. opt. value}|}{\max\{1, |\text{the lower bound for opt. value}|\}},$$

ϵ_{feas} = the maximum error in the equality constraints,
cpu : cpu time in sec. to solve an SDP relaxation problem.

alkyl.gms : a benchmark problem from globallib

$$\begin{array}{ll} \text{min} & -6.3x_5x_8 + 5.04x_2 + 0.35x_3 + x_4 + 3.36x_6 \\ \text{sub.to} & -0.820x_2 + x_5 - 0.820x_6 = 0, \\ & 0.98x_4 - x_7(0.01x_5x_{10} + x_4) = 0, \\ & -x_2x_9 + 10x_3 + x_6 = 0, \\ & x_5x_{12} - x_2(1.12 + 0.132x_9 - 0.0067x_9^2) = 0, \\ & x_8x_{13} - 0.01x_9(1.098 - 0.038x_9) - 0.325x_7 = 0.574, \\ & x_{10}x_{14} + 22.2x_{11} = 35.82, \\ & x_1x_{11} - 3x_8 = -1.33, \\ & \text{lbd}_i \leq x_i \leq \text{ubd}_i \ (i = 1, 2, \ldots, 14). \end{array}$$

			sparse			Lasserre's dense		
problem	n	r	ϵ_{obj}	ϵ_{feas}	cpu	ϵ_{obj}	ϵ_{feas}	cpu
alkyl	14	2	4.1e-03	2.7e-01	0.9	6.3e-06	1.8e-02	17.6
alkyl	14	3	5.6e-10	2.0e-08	6.9	—	—	—

$r =$ relaxation order,

$$\epsilon_{\text{obj}} = \frac{|\text{the lower bound for opt. value} - \text{the approx. opt. value}|}{\max\{1, |\text{the lower bound for opt. value}|\}},$$

$\epsilon_{\text{feas}} =$ the maximum error in the equality constraints,

cpu : cpu time in sec. to solve an SDP relaxation problem.

Some other benchmark problems from globallib

			sparse			Lasserre's dense		
problem	n	r	ϵ_{obj}	ϵ_{feas}	cpu	ϵ_{obj}	ϵ_{feas}	cpu
ex3_1_1	8	3	6.3e-09	4.7e-04	5.5	0.7e-08	2.5e-03	597.8
st_bpaf1b*	10	2	3.8e-08	2.8e-08	1.0	4.6e-09	7.2e-10	1.7
st_e07*	10	2	0.0e+00	8.1e-05	0.4	0.0e+00	8.8e-06	3.0
ex2_1_3	13	2	5.1e-09	3.5e-09	0.5	1.6e-09	1.5e-09	7.7
ex9_1_1	13	2	0.0	4.5e-06	1.5	0.0	9.2e-07	7.7
ex9_2_3*	16	2	0.0e+00	5.7e-06	2.3	0.0e+00	7.5e-06	49.7
ex2_1_8*	24	2	1.0e-05	0.0e+00	304.6	3.4e-06	0.0e+00	1946.6

$r =$ relaxation order,

$$\epsilon_{\text{obj}} = \frac{|\text{the lower bound for opt. value} - \text{the approx. opt. value}|}{\max\{1, |\text{the lower bound for opt. value}|\}},$$

$\epsilon_{\text{feas}} =$ the maximum error in the equality constraints,

cpu : cpu time in sec. to solve an SDP relaxation problem.

Some other benchmark problems from globallib

			sparse			Lasserre's dense		
problem	n	r	$\epsilon_{\rm obj}$	$\epsilon_{\rm feas}$	cpu	$\epsilon_{\rm obj}$	$\epsilon_{\rm feas}$	cpu
ex3_1_1	8	3	6.3e-09	4.7e-04	5.5	0.7e-08	2.5e-03	597.8
st_bpaf1b*	10	2	3.8e-08	2.8e-08	1.0	4.6e-09	7.2e-10	1.7
st_e07*	10	2	0.0e+00	8.1e-05	0.4	0.0e+00	8.8e-06	3.0
ex2_1_3	13	2	5.1e-09	3.5e-09	0.5	1.6e-09	1.5e-09	7.7
ex9_1_1	13	2	0.0	4.5e-06	1.5	0.0	9.2e-07	7.7
ex9_2_3*	16	2	0.0e+00	5.7e-06	2.3	0.0e+00	7.5e-06	49.7
ex2_1_8*	24	2	1.0e-05	0.0e+00	304.6	3.4e-06	0.0e+00	1946.6

- $\star$ — no tight optimal value before.
- The sparse relaxation attains approx. opt. solutions with the same quality as the dense relaxation.
- The sparse relaxation is much faster than the dense relaxation in large dim. and higher relaxation order cases.

Outline

1. POPs (Polynomial Optimization Problems)
2. Nonnegative polynomials and SOS (Sum of Squares) polynomials
3. SOS relaxation of unconstrained POPs
4. Conversion of SOS relaxation into an SDP (Semidefinite Program)
5. Structured sparsity
6. SOS relaxation of constrained POPs
7. Numerical results
8. Concluding remarks

- Lasserre's (dense) relaxation
  — theoretical convergence but expensive in practice.
- The proposed sparse relaxation
  = Lasserre's (dense) relaxation + sparsity
  — no theoretical convergence but very powerful in practice.

- There remain many issues to be studied further.
  - Exploiting sparsity.
  - Large-scale SDPs.

- sparse SOS and SDP relaxations will work as very powerful methods to compute global optimal solutions of POPs.