半正定値計画問題への招待

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ワークショップ「最適化理論の産業・諸科学への応用

九州大学伊都キャンパス

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5. 簡単な例
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SDP は LP の対称行列空間への拡張

LP: minimize $-X_{11} - 2X_{12} - 5X_{22}$

subject to $2X_{11} + 3X_{12} + X_{22} = 7,\ X_{11} + X_{12} \geq 1,$

$X_{11} \geq 0,\ X_{12} \geq 0,\ X_{22} \geq 0.$

SDP: minimize $-X_{11} - 2X_{12} - 5X_{22}$

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$X_{11} \geq 0,\ X_{12} \geq 0,\ X_{22} \geq 0,$

$\begin{pmatrix} X_{11} & X_{12} \\ X_{12} & X_{22} \end{pmatrix} \succeq \boldsymbol{O}$ (半正定値).

- 共通点：$X_{11},\ X_{12},\ X_{22}$ に関する線形目的関数
- 共通点：$X_{11},\ X_{12},\ X_{22}$ に関する線形等式，線形不等式条件

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$\begin{pmatrix} X_{11} & X_{12} \\ X_{12} & X_{22} \end{pmatrix} \succeq \boldsymbol{O}$ (半正定値).

- SDP: 対称行列変数 $\begin{pmatrix} X_{11} & X_{12} \\ X_{12} & X_{22} \end{pmatrix}$ に対する半正定値条件 $\Leftrightarrow$

$X_{11} \geq 0,\ X_{22} \geq 0,\ X_{11}X_{22} - X_{12}^2 \geq 0$ — 非線形

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$X_{11} \geq 0,\ X_{12} \geq 0,\ X_{22} \geq 0,$

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- LP の許容解集合（条件を満たす X_{11}, X_{12}, X_{22} の集合）は多面体集合 $\Rightarrow$ 有限個の頂点のいずれかで最小値が達成 $\Rightarrow$ シンプレックス法
- SDP の許容解集合は凸集合であるが多面体集合ではない

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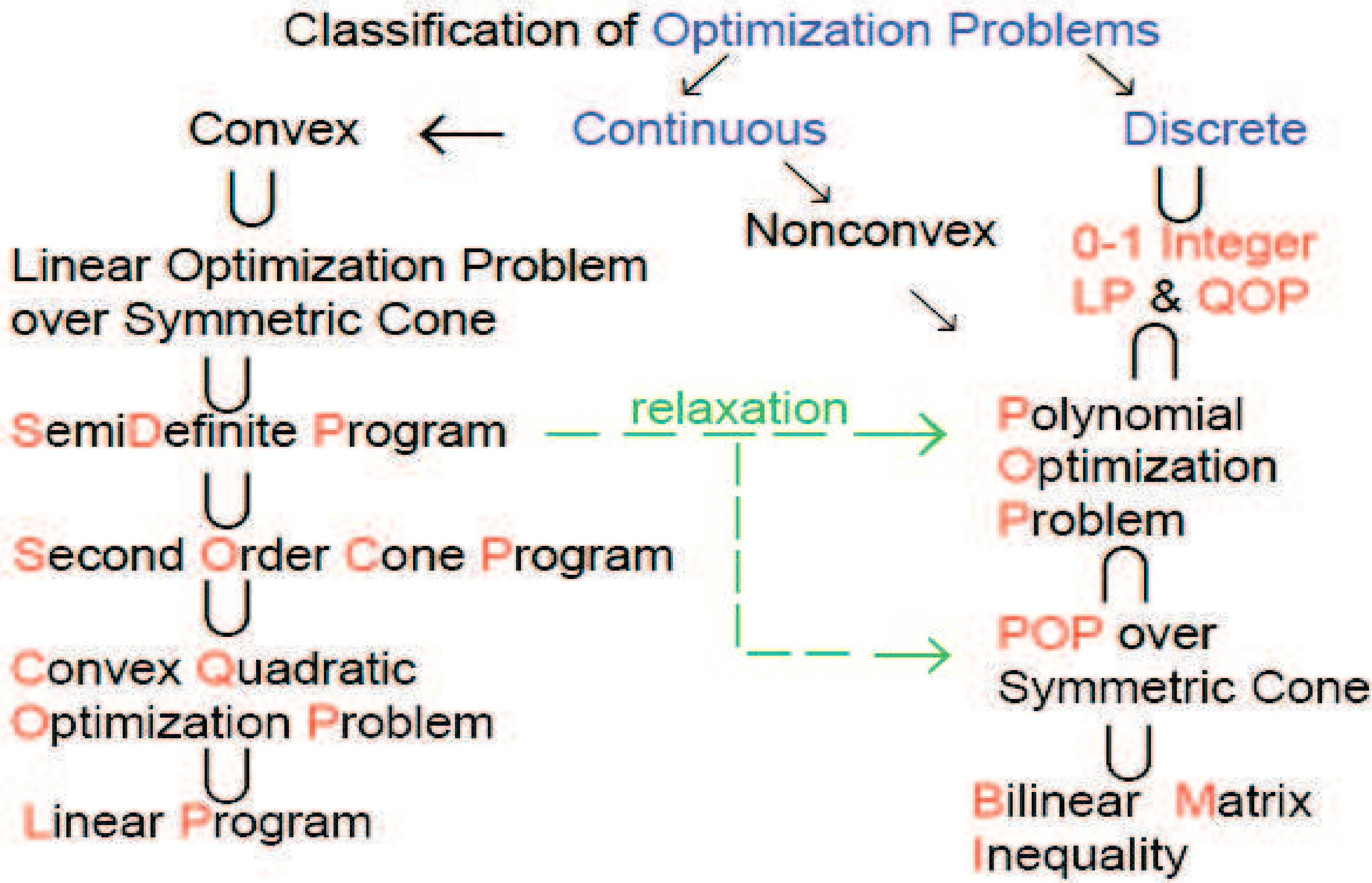
Classification of Optimization Problems
Convex
Continuous
Discrete
Nonconvex
Linear Optimization Problem
over Symmetric Cone
0-1 Integer
LP & QOP
SemiDefinite Program
relaxation
Polynomial
Optimization
Problem
Second Order Cone Program
POP over
Symmetric Cone
Convex Quadratic
Optimization Problem
Linear Program
Bilinear Matrix
Inequality

多くの分野へのさまざまな応用

- システムと制御 — 線形行列不等式 [6]
- 半正定値行列緩和
  - グラフの最大カット問題 [14]，最大クリーク問題
  - 線形，２次 $0-1$ 計画問題 [24]
  - 多項式最適化 [22, 35]
- ロバスト最適化 [4]
- 量子化学 [51]
- モーメント問題（応用確率論） [5, 23]
- . . .

サーベイ論文 — Todd [39] ,Vandenberghe-Boyd [45]
ハンドブック — Wolkowicz-Saigal-Vandenberghe [46]
ウェブサイト — Helmberg[15], Wolkowicz [47]

SDP を支える基礎理論

- Self-concordant 理論 [33]
- Euclidean Jordan 代数 [10, 36]
- 主双対内点法 [1, 17, 20, 27, 34]

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(LP)	minimize	$\boldsymbol{a}_0 \cdot \boldsymbol{x}$
	subject to	$\boldsymbol{a}_p \cdot \boldsymbol{x} = b_p \ (1 \leq p \leq m), \ \mathbb{R}^n \ni \boldsymbol{x} \geq \mathbf{0}$.

$\mathbb{R}$	:	実数の集合,
$\mathbb{R}^n$	:	n dim. ベクトルの線形空間 ,
$\boldsymbol{a}_p \in \mathbb{R}^n$	:	定数, n dim. ベクトル $(1 \leq p \leq m)$,
$b_p \in \mathbb{R}$	:	定数, 実数 $(1 \leq p \leq m)$,
$\boldsymbol{x} \in \mathbb{R}^n$	:	変数, n dim. ベクトル,
$\boldsymbol{a}_p \cdot \boldsymbol{x}$	$=$	$\sum_{i=1}^{n} [\boldsymbol{a}_p]_i x_i$ ($\boldsymbol{a}_p$ and $\boldsymbol{x}$ の内積).

(LP) minimize $\boldsymbol{a}_0 \cdot \boldsymbol{x}$

subject to $\boldsymbol{a}_p \cdot \boldsymbol{x} = b_p \ (1 \leq p \leq m), \ \mathbb{R}^n \ni \boldsymbol{x} \geq \mathbf{0}$.

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(SDP) minimize $\boldsymbol{A}_0 \bullet \boldsymbol{X}$

subject to $\boldsymbol{A}_p \bullet \boldsymbol{X} = b_p \ (1 \leq p \leq m),\ \mathbb{S}^n \ni \boldsymbol{X} \succeq \boldsymbol{O}$.

$\mathbb{S}^n$: $n \times n$ 対称行列の線形空間,

$\boldsymbol{A}_p \in \mathbb{S}^n$: 定数, $n \times n$ 対称行列 $(0 \leq p \leq m)$,

$b_p \in \mathbb{R}$: 定数, 実数 $(1 \leq p \leq m)$,

$\boldsymbol{X} \in \mathbb{S}^n$: $n \times n$ 変数, $n \times n$ 対称行列;

$$\boldsymbol{X} = (X_{ij}) = \begin{pmatrix} X_{11} & X_{12} & \ldots & X_{1n} \\ X_{21} & X_{22} & \ldots & X_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ X_{n1} & X_{n2} & \ldots & X_{nn} \end{pmatrix} \in \mathbb{S}^n,$$

$X_{ij} = X_{ji} \in \mathbb{R} \ (1 \leq i \leq j \leq n)$,

(LP) minimize $\boldsymbol{a}_0 \cdot \boldsymbol{x}$

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$\boldsymbol{X} \in \mathbb{S}^n_+ \quad \Leftrightarrow \quad \boldsymbol{X} \in \mathbb{S}^n$: 半正定値;

$\boldsymbol{X}$ の固有値が全て非負,

2 次形式$\boldsymbol{u}^T \boldsymbol{X} \boldsymbol{u} \geq 0, \ \forall \boldsymbol{u} \in \mathbb{R}^n$,

$\boldsymbol{X} \succeq \boldsymbol{O} \quad \Leftrightarrow \quad \boldsymbol{X} \in \mathbb{S}^n_+, \ \exists n$,

$\boldsymbol{A}_p \bullet \boldsymbol{X} \quad = \quad \sum_{i=1}^n \sum_{j=1}^n [\boldsymbol{A}_p]_{ij} \boldsymbol{X}_{ij}$ ($\boldsymbol{A}_p$ と $\boldsymbol{X}$ の内積).

$$\text{(LP)} \quad \text{minimize} \quad \boldsymbol{a}_0 \cdot \boldsymbol{x}$$
$$\text{subject to} \quad \boldsymbol{a}_p \cdot \boldsymbol{x} = b_p \ (1 \leq p \leq m), \ \mathbb{R}^n \ni \boldsymbol{x} \geq \mathbf{0}.$$

$$\text{(SDP)} \quad \text{minimize} \quad \boldsymbol{A}_0 \bullet \boldsymbol{X}$$
$$\text{subject to} \quad \boldsymbol{A}_p \bullet \boldsymbol{X} = b_p \ (1 \leq p \leq m), \ \mathbb{S}^n \ni \boldsymbol{X} \succeq \boldsymbol{O}.$$

$$\Uparrow \quad \left\{ \begin{array}{l} m = 2, \ n = 2, \ b_1 = 7, \ b_2 = 9, \\ \boldsymbol{X} = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}, \ \boldsymbol{A}_0 = \begin{pmatrix} -1 & -1 \\ -1 & -5 \end{pmatrix}, \\ \boldsymbol{A}_1 = \begin{pmatrix} 2 & 1.5 \\ 1.5 & 1 \end{pmatrix}, \ \boldsymbol{A}_2 = \begin{pmatrix} 2 & 0.5 \\ 0.5 & 3 \end{pmatrix}. \end{array} \right.$$

$$\text{minimize} \quad -X_{11} - 2X_{12} - 5X_{22}$$
$$\text{subject to} \quad 2X_{11} + 3X_{12} + X_{22} = 7, \ 2X_{11} + X_{12} + 3X_{22} = 9,$$
$$\begin{pmatrix} X_{11} & X_{12} \\ X_{12} & X_{22} \end{pmatrix} \succeq \boldsymbol{O}.$$

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等式標準形 (SDP):

$$\begin{array}{ll} \text{min.} & \boldsymbol{A}_0 \bullet \boldsymbol{X} \\ \text{sub.to} & \boldsymbol{A}_p \bullet \boldsymbol{X} = b_p \ (1 \leq p \leq m), \ \mathbb{S}^n \ni \boldsymbol{X} \succeq \boldsymbol{O}. \end{array}$$

$\Uparrow$?

システムと制御からの SDP の例 (SDP)':

$$\begin{array}{ll} \text{min} & \lambda \\ \text{sub. to} & \begin{pmatrix} \boldsymbol{X}\boldsymbol{A} + \boldsymbol{A}^T\boldsymbol{X} + \boldsymbol{C}^T\boldsymbol{C} & \boldsymbol{X}\boldsymbol{B} + \boldsymbol{C}^T\boldsymbol{D} \\ \boldsymbol{B}^T\boldsymbol{X} + \boldsymbol{D}^T\boldsymbol{C} & \boldsymbol{D}^T\boldsymbol{D} - \boldsymbol{I} \end{pmatrix} \preceq \lambda \boldsymbol{I}, \\ & \boldsymbol{X} \succeq -\lambda \boldsymbol{I}. \end{array}$$

$\boldsymbol{X} \in \mathbb{S}^n$, $\lambda \in \mathbb{R}$ 変数, $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$ 定数行列, $\boldsymbol{I}$: 単位行列

- (SDP)' は等式標準形 (SDP) に変換出来るか?
- 理論的には "Yes" $\Longleftarrow$ $\boldsymbol{X} \in \mathbb{S}^n$ の任意の実線形関数は $\boldsymbol{A} \bullet \boldsymbol{X} = \sum_{i=1}^n \sum_{j=1}^n \boldsymbol{A}_{ij} \boldsymbol{X}_{ij}$, $\exists \boldsymbol{A} \in \mathbb{S}^n$ と表現できる.
- 汎用ソフトウェアを適用するには自由変数を含む等式標準形または等式条件付き LMI 標準形への変換が必要

一般の SDP:

min.	$x_1, \ldots, x_k$ と $\boldsymbol{X}^q$ $(1 \leq q \leq t)$ の線形関数
sub.to	$x_1, \ldots, x_k$ と $\boldsymbol{X}^q$ $(1 \leq q \leq t)$ の線形等式,
	$x_1, \ldots, x_k$ と $\boldsymbol{X}^q$ $(1 \leq q \leq t)$ の線形 (行列) 不等式,
	$x_1, \ldots, x_k \in \mathbb{R}$ (自由変数),
	$\boldsymbol{X}^q \succeq \boldsymbol{O}$ $(1 \leq q \leq t)$ (半正定値行列変数).

線形行列不等式

$$\boldsymbol{F}(x_1, \ldots, x_k, \boldsymbol{X}_1, \ldots, \boldsymbol{X}_q) - \boldsymbol{B} \succeq \boldsymbol{O},$$

$\boldsymbol{F} : (x_1, \ldots, x_k, \boldsymbol{X}_1, \ldots, \boldsymbol{X}_q) \to \boldsymbol{Y} \in \mathbb{S}^m$: 線形，$\boldsymbol{B} \in \mathbb{S}^m$: 定数

$m = 1$ のときは，通常の線形不等式.

実非負変数は 1×1 半正定値対称行列変数とみなす.

$k \times m$ 行列変数 $\boldsymbol{U}$ 自由変数 U_{ij} の集まりとみなす.

例

$$\min \quad x + \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \bullet \boldsymbol{X} \quad \text{sub.to} \quad \begin{pmatrix} \boldsymbol{X} & \begin{matrix} 2 \\ 1 \end{matrix} \\ \begin{matrix} 2 & 1 \end{matrix} & x \end{pmatrix} \succeq \boldsymbol{O}.$$

$\Longrightarrow$ LMI 標準形（等式標準形の双対問題）

$$\min \quad x + 2y_1 + 2y_2 + 3y_3$$

$$\text{sub.to} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} y_1 + \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} y_2 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} y_3$$

$$+ \begin{pmatrix} \boldsymbol{O} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{pmatrix} x + \begin{pmatrix} \boldsymbol{O} & \begin{matrix} 2 \\ 1 \end{matrix} \\ \begin{matrix} 2 & 1 \end{matrix} & 0 \end{pmatrix} \succeq \boldsymbol{O}.$$

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対称行列 A の固有値

$$\begin{aligned}\text{最大固有値} &= \min\{\lambda : \lambda \boldsymbol{I} \succeq \boldsymbol{A}\} \\ &= \min\{\lambda : \lambda \boldsymbol{I} - \boldsymbol{A} \succeq \boldsymbol{O}\}. \\ \text{最小固有値} &= \max\{\lambda : \boldsymbol{A} - \lambda \boldsymbol{I} \succeq \boldsymbol{O}\}.\end{aligned}$$

- A 線形行列不等式 (LMI) $\boldsymbol{A}(\cdot) \succeq \boldsymbol{O}$, ただし $\boldsymbol{A}(\cdot)$ は行列，ベクトルに関する線形写像，は

$$\text{maximize } \lambda \text{ subject to } \boldsymbol{A}(\cdot) - \lambda \boldsymbol{I} \succeq \boldsymbol{O}.$$

と定式化できる．例えば，

$$\boldsymbol{A}(\boldsymbol{X}) = \begin{pmatrix} \boldsymbol{X}\boldsymbol{A} + \boldsymbol{A}^T\boldsymbol{X} + \boldsymbol{C}^T\boldsymbol{C} & \boldsymbol{X}\boldsymbol{B} + \boldsymbol{C}^T\boldsymbol{D} \\ \boldsymbol{B}^T\boldsymbol{X} + \boldsymbol{D}^T\boldsymbol{C} & \boldsymbol{D}^T\boldsymbol{D} - \boldsymbol{I} \end{pmatrix} \succeq \boldsymbol{O}.$$

参考文献

[6] S. Boyd, L. El Ghaui, E. Feron and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, SIAM, Philadelphia, 1994.

Matrix approximation problem — 1

$\boldsymbol{F}_p$ を $k \times \ell$ 行列とする $(0 \le p \le m)$. $\boldsymbol{F}_0$ を $\boldsymbol{F}_p$ $(1 \le p \le m)$ の線形結合で近似せよ;

$$\text{minimize } \{\|\boldsymbol{F}(\boldsymbol{x})\| : \boldsymbol{x} \in \mathbb{R}^m\},$$

ただし, $\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{F}_0 - \sum_{p=1}^m \boldsymbol{F}_p x_p$ for $\forall \boldsymbol{x} = (x_1, \ldots, x_m)^T$.

- この問題の最適化問題への定式化は norm の取り方に依存する

$$\|\boldsymbol{A}\|_\infty = \max\{|A_{ij}| : 1 \le i \le k,\ 1 \le j \le \ell\}$$ (∞ norm) $\Rightarrow$ LP

$$\|\boldsymbol{A}\|_F = \left(\sum_{i=1}^k \sum_{j=1}^\ell A_{ij}^2\right)^{1/2}$$ (Frobenius norm) $\Rightarrow$ convex QP

$$\|\boldsymbol{A}\|_{L_2} = \max_{\|\boldsymbol{u}\|_2=1} \|\boldsymbol{A}\boldsymbol{u}\| = \left(\boldsymbol{A}^T\boldsymbol{A} \text{ の最大固有値}\right)^{1/2}$$

(L_2 operator norm) $\Rightarrow$ SDP

Matrix approximation problem — 2

$\boldsymbol{F}_p$ を $k \times \ell$ 行列とする $(0 \leq p \leq m)$. $\boldsymbol{F}_0$ を $\boldsymbol{F}_p$ $(1 \leq p \leq m)$ の線形結合で近似せよ;

$$\text{minimize } \{\|\boldsymbol{F}(\boldsymbol{x})\| : \boldsymbol{x} \in \mathbb{R}^m\},$$

ただし, $\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{F}_0 - \sum_{p=1}^{m} \boldsymbol{F}_p x_p$ for $\forall \boldsymbol{x} = (x_1, \ldots, x_m)^T$.

Matrix approximation problem — 2

$\boldsymbol{F}_p$ を $k \times \ell$ 行列とする $(0 \le p \le m)$. $\boldsymbol{F}_0$ を $\boldsymbol{F}_p$ $(1 \le p \le m)$ の線形結合で近似せよ;

$$\text{minimize } \{\|\boldsymbol{F}(\boldsymbol{x})\| : \boldsymbol{x} \in \mathbb{R}^m\},$$

ただし, $\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{F}_0 - \sum_{p=1}^m \boldsymbol{F}_p x_p$ for $\forall \boldsymbol{x} = (x_1, \dots, x_m)^T$.

$$\|\boldsymbol{A}\|_{L_2} = \max_{\|\boldsymbol{u}\|_2=1} \|\boldsymbol{A}\boldsymbol{u}\| = \left(\boldsymbol{A}^T\boldsymbol{A}\text{ の最大固有値}\right)^{1/2}$$

(L_2 operator norm) $\Rightarrow$ SDP

$$\text{minimize } \left\{\|\boldsymbol{F}(\boldsymbol{x})\|_{L_2} : \boldsymbol{x} \in \mathbb{R}^m\right\}$$

$$\Downarrow$$

minimize "$\boldsymbol{F}(\boldsymbol{x})^T\boldsymbol{F}(\boldsymbol{x})$ の最大固有値"

$$\Downarrow$$

minimize λ subject to $\lambda \boldsymbol{I} - \boldsymbol{F}(\boldsymbol{x})^T\boldsymbol{F}(\boldsymbol{x}) \succeq \boldsymbol{O}$

$\Downarrow$ Schur complement

minimize λ subject to $\begin{pmatrix} \boldsymbol{I} & \boldsymbol{F}(\boldsymbol{x}) \\ \boldsymbol{F}(\boldsymbol{x})^T & \lambda \boldsymbol{I} \end{pmatrix} \succeq \boldsymbol{O}$ (SDP)

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LP の主問題と双対問題

(P)	min	$\boldsymbol{a}_0 \cdot \boldsymbol{x}$	s.t.	$\boldsymbol{a}_p \cdot \boldsymbol{x} = b_p \ (\forall p), \ \boldsymbol{x} \geq \mathbf{0}$.
(D)	max	$\sum_{p=1}^{m} b_p y_p$	s.t.	$\sum_{p=1}^{m} \boldsymbol{a}_p y_p + s = \boldsymbol{a}_0, \ \mathbb{R}^n \ni s \geq \mathbf{0}$.

弱双対性:

LP の主問題と双対問題

(P)	min	$\boldsymbol{a}_0 \cdot \boldsymbol{x}$	s.t.	$\boldsymbol{a}_p \cdot \boldsymbol{x} = b_p \ (\forall p), \ \boldsymbol{x} \geq \mathbf{0}$.
(D)	max	$\sum_{p=1}^{m} b_p y_p$	s.t.	$\sum_{p=1}^{m} \boldsymbol{a}_p y_p + s = \boldsymbol{a}_0, \ \mathbb{R}^n \ni s \geq \mathbf{0}$.

弱双対性:

$$\text{LP} \quad : \quad \boldsymbol{x} \cdot s = \boldsymbol{a}_0 \cdot \boldsymbol{x} - \sum_{j=1}^{m} b_p y_p \geq 0, \ \forall \text{ 許容解 } (\boldsymbol{x}, \ \boldsymbol{y}, \ s).$$

$$\text{SDP} \quad : \quad \boldsymbol{X} \bullet S = \boldsymbol{A}_0 \bullet \boldsymbol{X} - \sum_{j=1}^{m} b_p y_p \geq 0, \ \forall \text{ 許容解 } (\boldsymbol{X}, \ \boldsymbol{y}, \ \boldsymbol{S}).$$

SDP の主問題と双対問題

(P)	min.	$\boldsymbol{A}_0 \bullet \boldsymbol{X}$	sub.to	$\boldsymbol{A}_p \bullet \boldsymbol{X} = b_p \ (\forall p), \ \boldsymbol{X} \succeq \boldsymbol{O}$.
(D)	max.	$\sum_{p=1}^{m} b_p y_p$	sub.to	$\sum_{p=1}^{m} \boldsymbol{A}_p y_p + \boldsymbol{S} = \boldsymbol{A}_0, \ \boldsymbol{S} \succeq \boldsymbol{O}$.

LP の主問題と双対問題

(P)	min	$\boldsymbol{a}_0 \cdot \boldsymbol{x}$	s.t.	$\boldsymbol{a}_p \cdot \boldsymbol{x} = b_p \ (\forall p), \ \boldsymbol{x} \geq \mathbf{0}$.
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強双対性:

LP の主問題と双対問題

(P)	min	$\boldsymbol{a}_0 \cdot \boldsymbol{x}$	s.t.	$\boldsymbol{a}_p \cdot \boldsymbol{x} = b_p \ (\forall p), \ \boldsymbol{x} \geq \mathbf{0}.$
(D)	max	$\sum_{p=1}^m b_p y_p$	s.t.	$\sum_{p=1}^m \boldsymbol{a}_p y_p + s = \boldsymbol{a}_0, \ \mathbb{R}^n \ni s \geq \mathbf{0}.$

強双対性:

∃ 許容解 $(\boldsymbol{x}, \ \boldsymbol{y}, \ s)$ $(\boldsymbol{x} \geq 0, \ \boldsymbol{y} \geq \mathbf{0})$ $\Rightarrow$

$$\text{LP} \quad : \quad \bar{\boldsymbol{x}} \cdot \bar{s} = \boldsymbol{a}_0 \cdot \boldsymbol{x} - \sum_{j=1}^{m} b_p \bar{y}_p = 0, \ \forall \text{ 最適解 } (\bar{\boldsymbol{x}}, \ \bar{\boldsymbol{y}}, \ \bar{s}).$$

∃ 内点許容解 $(\boldsymbol{X}, \ \boldsymbol{y}, \ \boldsymbol{S})$ $(\boldsymbol{X} \succ \boldsymbol{O}, \ \boldsymbol{S} \succ \boldsymbol{O})$ $\Rightarrow$

$$\text{SDP} \quad : \quad \bar{\boldsymbol{X}} \bullet \bar{\boldsymbol{S}} = \boldsymbol{A}_0 \bullet \bar{\boldsymbol{X}} - \sum_{j=1}^{m} b_p \bar{y}_p = 0, \ \forall \text{ 最適解 } (\bar{\boldsymbol{X}}, \ \bar{\boldsymbol{y}}, \ \bar{\boldsymbol{S}}).$$

SDP の主問題と双対問題

(P)	min.	$\boldsymbol{A}_0 \bullet \boldsymbol{X}$	sub.to	$\boldsymbol{A}_p \bullet \boldsymbol{X} = b_p \ (\forall p), \ \boldsymbol{X} \succeq \boldsymbol{O}.$
(D)	max.	$\sum_{p=1}^m b_p y_p$	sub.to	$\sum_{p=1}^m \boldsymbol{A}_p y_p + \boldsymbol{S} = \boldsymbol{A}_0, \ \boldsymbol{S} \succeq \boldsymbol{O}.$

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4. 一般の SDP
5. 簡単な例
6. 双対性
7. 数値解法
8. センサーネットワークの位置推定問題への応用
   — 半正定値計画緩和の予告編 —

SDP に対する数値解法

- 主双対内点法 (Primal-Dual Interior-point methods)
  CSDP(Borchers[7]),
  SDPA(Fujisawa-K-Nakata-Yamashita[49]),
  SDPT3(Toh-Todd-Tutuncu[42]), SeDuMi(Sturm[37])
- Dual scaling, DSDP(Benson-Ye-Zhang[3])
- その他の非線形最適化
  - Spectral bundle method(Helmberg-Rendl[17])
  - Gradient-based log-barrier method(Burer-Monteiro[9])
  - PENON(Kocvara [19]) — Augmented Lagrangian
  - Saddle point mirror-prox algorithm (Lu-Nemirovski-Monteiro[26])

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- 中規模 (e.g. $n, \ m \leq 5000$)， 高精度.
- 大規模 (e.g., $n, m \geq 10,000$)， 低精度.

- 並列計算:

SDPA $\Rightarrow$ SDPARA(Y-F-K[49]), SDPARA-C(N-Y-F-K[31])
DSDP $\Rightarrow$ PDSDP(Benson[2]), CSDP $\Rightarrow$ Borchers-Young[8]
Spectral bundle method $\Rightarrow$ Nayakkankuppam[32]

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Sensor network localization 問題: $s = 2$ or 3.

$$\boldsymbol{x}^p \in \mathbb{R}^s \quad : \quad \text{sensor の位置, 座標 (未知)} \ (p = 1, 2, \ldots, m),$$

$$\boldsymbol{x}^r = \boldsymbol{a}^r \in \mathbb{R}^s \quad : \quad \text{ancho の位置, 座標 (既知)} \ (r = m+1, \ldots, n),$$

$$d_{pq}^2 \quad = \quad \|\boldsymbol{x}^p - \boldsymbol{x}^q\|^2 + \epsilon_{pq} \ \text{— 距離 (既知) for} \ (p, q) \in \mathcal{N},$$

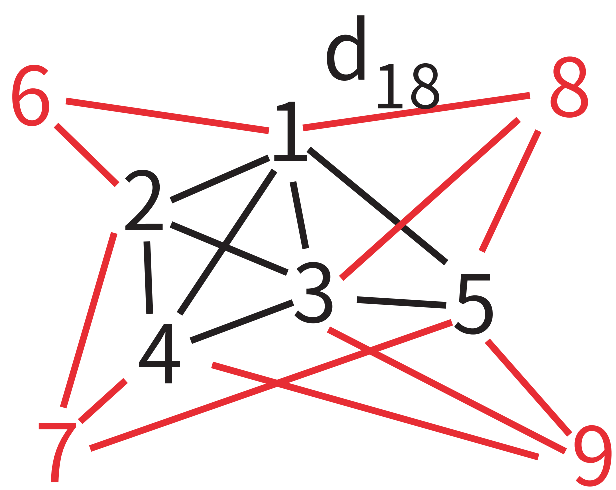
$$\mathcal{N} \quad = \quad \{(p, q) : \|\boldsymbol{x}^p - \boldsymbol{x}^q\| \leq \rho = \text{a radio range (既知)}\}$$

ここで ϵ_{pq} : noise

$m = 5, \ n = 9$.

$1, \ldots, 5$: sensor

$6, 7, 8, 9$: anchor



anchor の位置は既知

∀ edge に対して距離が既知

sensor の位置を計算せよ

- $\epsilon_{pq} = 0 \Rightarrow$ 多変数２次方程式系
  $\|\boldsymbol{x}^p - \boldsymbol{x}^q\|^2 = d_{pq}^2 \ ((p, q) \in \mathcal{N})$,
  $\boldsymbol{x}^r = \boldsymbol{a}^r \ (r = m+1, \ldots, n)$
- $\epsilon_{pq} \neq 0 \Rightarrow$ "誤差"の最小化

minimize $\displaystyle\sum_{(p,q)\in\mathcal{N}} \left| \|\boldsymbol{x}^p - \boldsymbol{x}^q\|^2 - d_{pq}^2 \right|$

- いずれの場合も半正定値計画緩和（≈ 近似）が適用出来る

m = sensor の個数, $[0,1]^2$ 上に random に分布
4 つの anchor $[0,1]^2$ の 4 隅に配置
ρ =radio range $= 0.1$, noise $\epsilon_{pq} = 0$

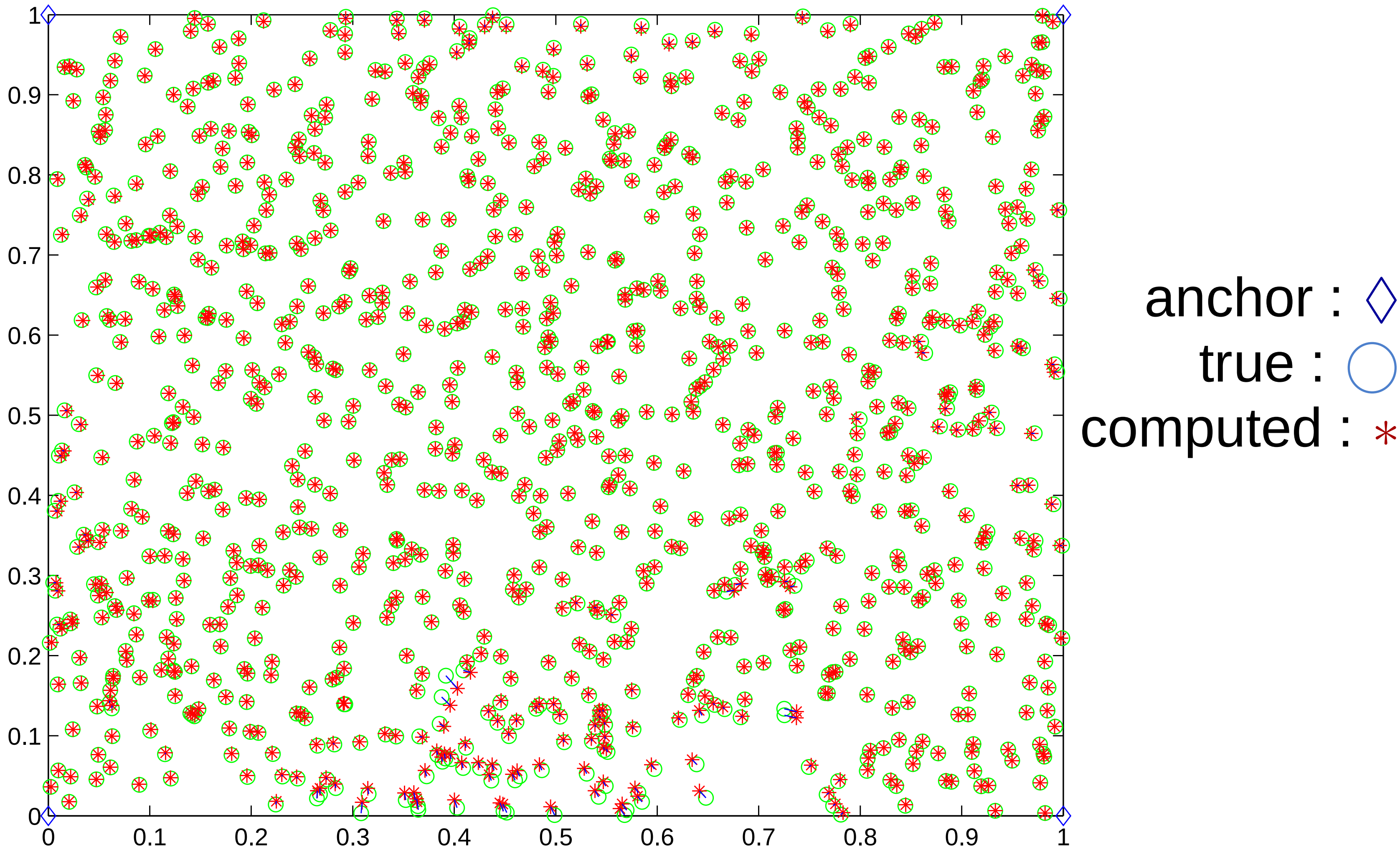
定式化: $\mathcal{N} = \{(p,q) : 1 \le p < q \le m,\ \|\boldsymbol{x}^p - \boldsymbol{x}^q\| < \rho\}$,
$\|\boldsymbol{x}^p - \boldsymbol{x}^q\|^2 = d_{pq}^2\ (p,q) \in \mathcal{N}$ — 多変数 2 次方程式系,
$\boldsymbol{x}^r = \boldsymbol{a}^r\ (r = m+1, \ldots, m+4)$

$\Rightarrow$ SDP に緩和（$\approx$ 近似）

m	1000	2000	4000
計算時間	11.7 秒	25.3 秒	41.9 秒
rmsd	4.8e-05	1.5e-05	2.6e-05

- root mean square distance = $\left(\frac{1}{m}\sum_{i=1}^{m} \|\boldsymbol{x}^p - \bar{\boldsymbol{x}}^p(\text{真値})\|^2\right)^{1/2}$
- SDP は SDPA を用いて解いた．CPU = 2.7GHz Intel Core i7

m = 1000 sensors, noise $\epsilon_{pq} = 0$

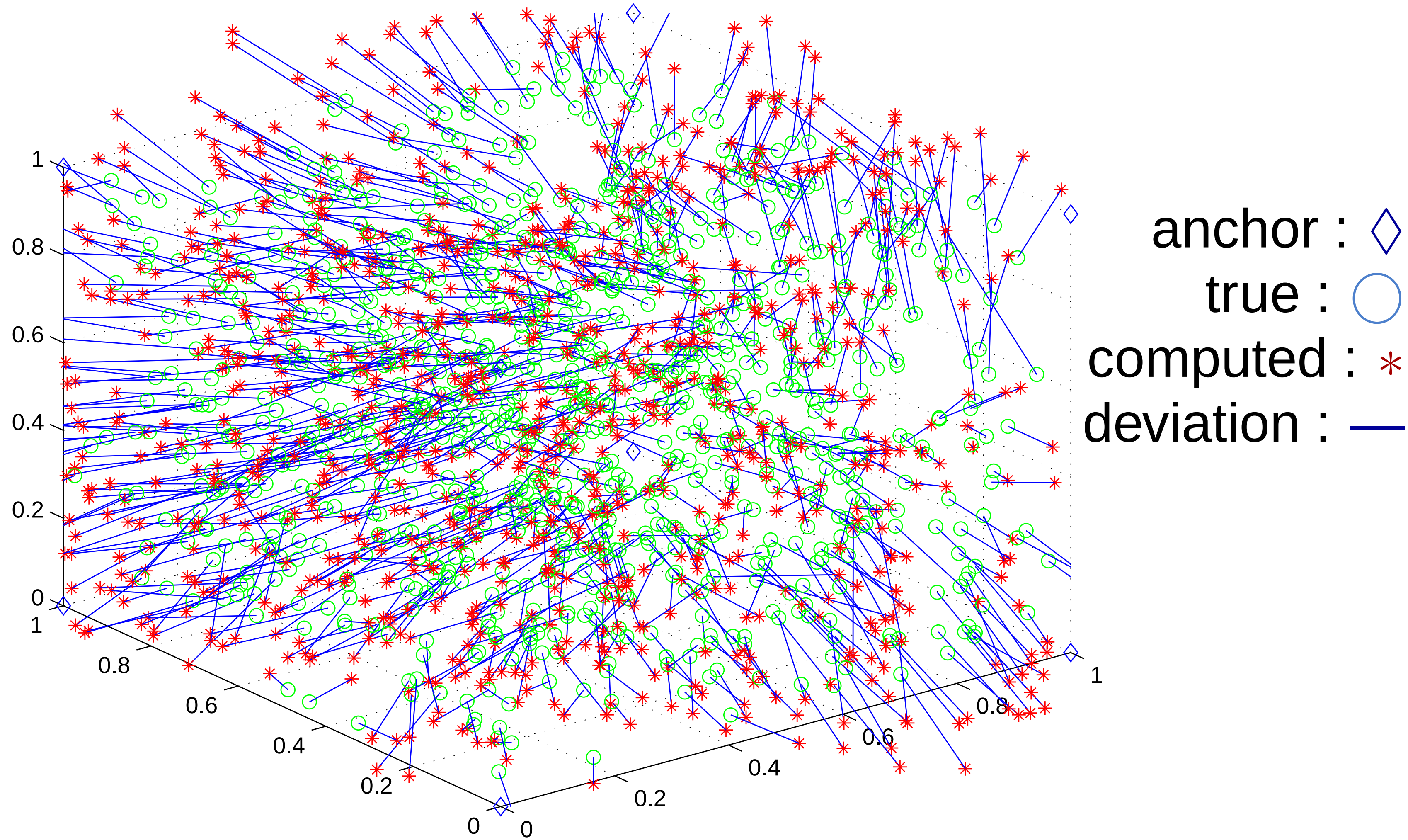


3 dim, 1000 sensors, radio range = 0.3, noise $\epsilon_{pq} \leftarrow N(0, 0.1)$;

(誤差を含んだ距離)$\hat{d}_{pq} = (1 + \epsilon_{pq})d_{pq}$(真の距離)

定式化：多変数の２次等式条件の下での線形目的関数の最小化

⇒ SDP に緩和（≈ 近似）　　計算時間 82.9 秒



O : Sensor true locations vs * : the ones computed by SFSDP

3 dim, 1000 sensors, radio range = 0.3, noise $\epsilon_{pq} \leftarrow N(0, 0.1)$;

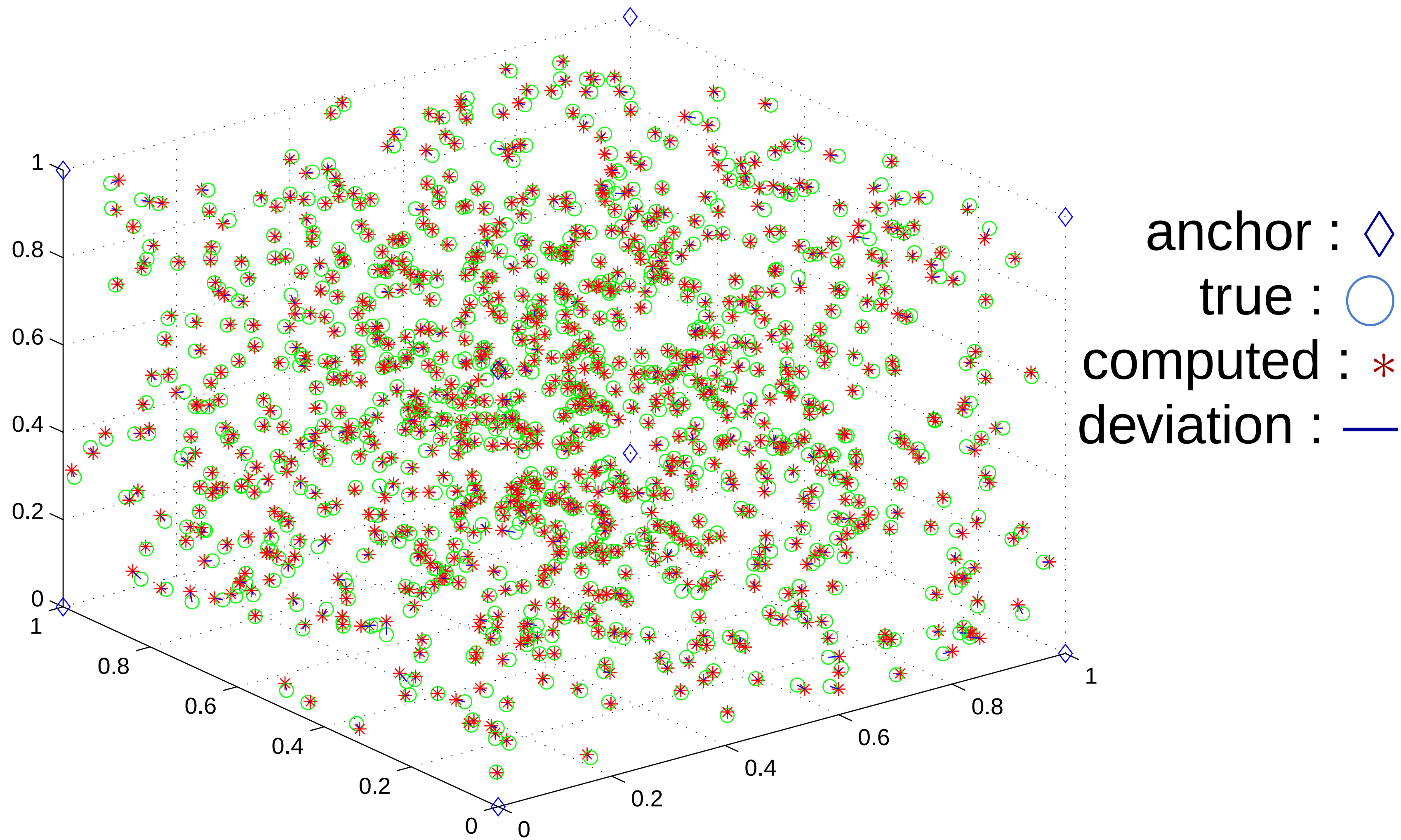
(誤差を含んだ距離)$\hat{d}_{pq} = (1 + \epsilon_{pq}) d_{pq}$(真の距離)

定式化：多変数の２次等式条件の下での線形目的関数の最小化

3 dim, 1000 sensors, radio range = 0.3, noise $\epsilon_{pq} \leftarrow N(0, 0.1)$;

(誤差を含んだ距離)$\hat{d}_{pq} = (1 + \epsilon_{pq}) d_{pq}$(真の距離)

定式化：多変数の２次等式条件の下での線形目的関数の最小化
⇒ SDP に緩和（≈ 近似）+最急降下法　　計算時間 112.0 秒



O : Sensor true locations vs * : the ones computed by SFSDP

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