Semidefinite Programming Relaxation and Lagrangian Relaxation for Polynomial Optimization Problems

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- Main purpose of my talk is "an introduction to the recent development of SDP relaxation in connection with the classical Lagrangian relaxation".

- Although the title includes "polynomial optimization problems", I will mainly talk about "quadratic optimization problems" for simplicity of discussions.
- But most of the discussions can be extended to "polynomial optimization problems".
- This material is available at http://www.is.titech.ac.jp/∼kojima/talk.html

Outline

1. **Optimization problems and their relaxation**
2. **Lagrangian relaxation**
3. **Lagrangian dual**
4. **SDP* relaxation of QOPs (quadratic optimization problems)**
5. **Lagrangian relaxation = SDP relaxation for QOPs**
6. **Summary**

$\star$ **: Semidefinite Program**

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1. **Optimization problems and their relaxation**
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$\star$: **Semidefinite Program**

Optimization Problem

$$\mathcal{P}_0 \ \text{minimize} \ \ f_0(x) \ \text{sub. to} \ x \in S_0, \ \text{where} \ f_0 : \mathbb{R}^n \rightarrow \mathbb{R} \ \text{and} \ S_0 \subset \mathbb{R}^n.$$

Difficult to compute exact global optimal solutions of general nonlinear programs and combinatorial optimization problems

Equality and inequality constrained optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0 \ (i = 1, 2, \ldots, \ell), \ f_j(x) = 0 \ (j = \ell + 1, \ldots, m). \end{array}$$

- Various assumptions imposed on f_i

"Continuous", "Smooth", "Convex"

"Linear + Quadratic", "Multivariate polynomial functions"

Optimization Problem

$$\mathcal{P}_0 \text{ minimize } f_0(x) \text{ sub. to } x \in S_0, \text{ where } f_0 : \mathbb{R}^n \to \mathbb{R} \text{ and } S_0 \subset \mathbb{R}^n.$$

Difficult to compute exact global optimal solutions of general nonlinear programs and combinatorial optimization problems

Equality and inequality constrained optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0 \ (i = 1, 2, \ldots, \ell), \ f_j(x) = 0 \ (j = \ell + 1, \ldots, m). \end{array}$$

- "Linear + Quadratic" is easily manageable, yet has enough power to describe various optimization models including combinatorial optimization problem;

$$\text{0-1 variable; } x_j = 0 \text{ or } 1 \Leftrightarrow x_j(x_j - 1) = 0 \quad \text{(quadratic equality)}$$

- Powerful mathematics and tools behind "Linear + Quadratic", "Multivariate polynomial functions" such as SDP relaxation and sums of squares polynomial relaxation.

Optimization Problem

$$\mathcal{P}_0 \ \text{minimize} \ \ f_0(x) \ \text{sub. to} \ x \in S_0, \ \text{where} \ f_0 : \mathbb{R}^n \rightarrow \mathbb{R} \ \text{and} \ S_0 \subset \mathbb{R}^n.$$

Example 1 QOP (Quadratic optimization problem)

$$\text{minimize} \ \ x_2^2 \ \text{sub.to} \ x_1^2 + x_2^2 \leq 4, \ -x_1^2/8 + 1 \leq x_2.$$

Example 2 POP (Polynomial optimization problem)

$$\text{minimize} \ \ -x_1^3 + 2x_1x_2^2 \ \text{sub.to} \ x_1^4 + x_2^4 \leq 1, \ x_1 \geq 0, \ x_1^2 + x_2^2 \geq 0.5.$$

We will mainly focus our attention to QOPs, but we can adapt the discussions here to POPs with slight modification.

Optimization Problem

$\mathcal{P}_0$ **minimize** $f_0(x)$ **sub. to** $x \in S_0$, **where** $f_0 : \mathbb{R}^n \rightarrow \mathbb{R}$ **and** $S_0 \subset \mathbb{R}^n$.

$\Downarrow$

Approximation of global optimal solutions:

(i) Methods to generate a feasible solution $x \in S_0$ **having a smaller objective value** $f_0(x)$.

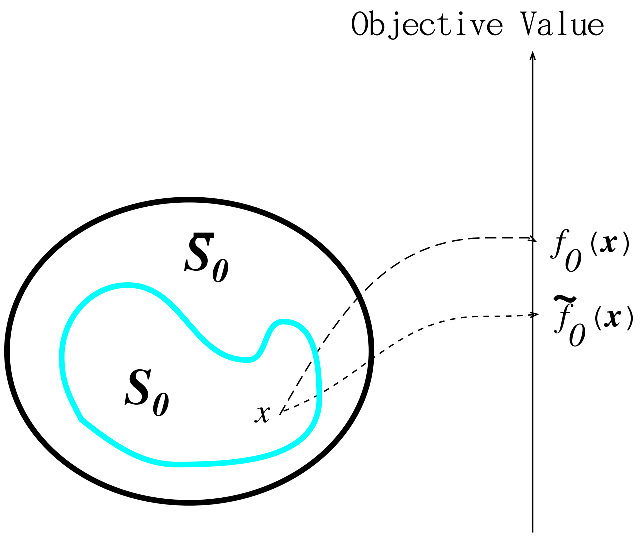
(ii) Methods to compute a lower bound for the unknown optimal value.

(ii) $\Longleftarrow$ **Various relaxation techniques**

Optimization Problem

$\mathcal{P}_0$ **minimize** $f_0(x)$ **sub. to** $x \in S_0$, **where** $f_0 : \mathbb{R}^n \to \mathbb{R}$ **and** $S_0 \subset \mathbb{R}^n$.

Relaxation of $\mathcal{P}_0$: $\widetilde{\mathcal{P}}_0$ **minimize** $\tilde{f}_0(x)$ **sub.to** $x \in \tilde{S}_0$,
where $S_0 \subseteq \tilde{S}_0$, **and** $\tilde{f}_0(x) \le f_0(x)$ $(\forall x \in S_0)$



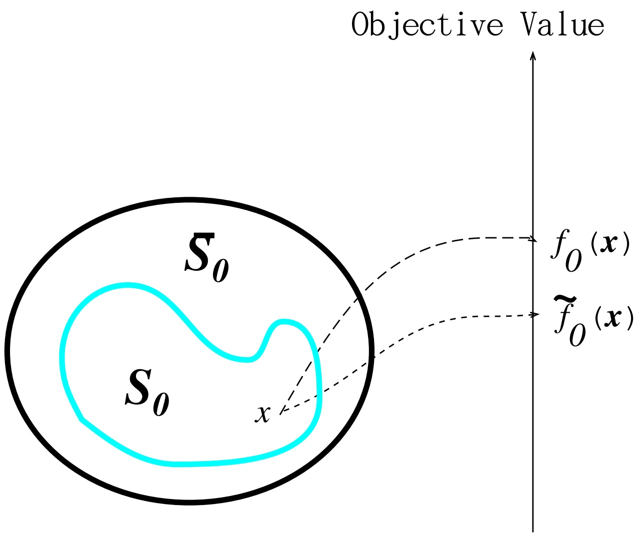
$f_0^* \equiv$ **the unknown min. value of** $\mathcal{P}_0 \ge \tilde{f}_0^* \equiv$ **the min. value of** $\widetilde{\mathcal{P}}_0$

If the difference $f_0(\widehat{x}) - \tilde{f}_0^*$ **between** $f_0(\widehat{x})$ **at a feasible solution** $\widehat{x} \in S_0$ **and** $\tilde{f}_0^*$ **is small, then we use** $\widehat{x}$ **as an approximate optimal solution of** $\mathcal{P}_0$

Optimization Problem

$$\mathcal{P}_0 \ \text{minimize} \ \ f_0(x) \ \text{sub. to} \ x \in S_0, \ \text{where} \ f_0 : \mathbb{R}^n \longrightarrow \mathbb{R} \ \text{and} \ S_0 \subset \mathbb{R}^n.$$

**Relaxation of $\mathcal{P}_0$: $\widetilde{\mathcal{P}}_0$ minimize $\tilde{f}_0(x)$ sub.to $x \in \tilde{S}_0$,
where $S_0 \subseteq \tilde{S}_0$, and $\tilde{f}_0(x) \leq f_0(x)$ $(\forall x \in S_0)$**



Conditions to be satisfied by the relaxation problem $\widetilde{\mathcal{P}}_0$:

- $S_0 \subseteq \tilde{S}_0$
- $\tilde{f}_0(x) \leq f_0(x)$ $(\forall x \in S_0)$
- For $y \notin S_0$, $\tilde{f}_0(y)$ can take any value

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$\star$: Semidefinite Programming

Lagrangian relaxation — A classical method of constructing relaxations of equality and/or inequality constrained optimization problems

Inequality constrained optimization problem

$$\text{minimize} \quad f_0(x) \text{ sub.to } x \in S_0 = \{x \in \mathbb{R}^n : f_j(x) \leq 0 \ (j = 1, \ldots, m)\}$$

Lagrangian function:

$$L(x, w) = f_0(x) + w_1 f_1(x) + w_2 f_2(x) + \cdots + w_m f_m(x),$$

where $w \in \mathbb{R}^m_+ \equiv \{w = (w_1, w_2, \ldots, w_m) \in \mathbb{R}^m : w_j \geq 0\}$.

Properties of Lagrangian function: for $\forall w \in \mathbb{R}^m_+$,

$$x \in S_0 \Rightarrow f_j(x) \leq 0 \ (j = 1, 2, \ldots, m) \Rightarrow$$

$$L(x, w) = f_0(x) + w_1 f_1(x) + w_2 f_2(x) + \cdots + w_m f_m(x) \leq f_0(x)$$

Lagrange relaxation problem: For $\forall$ fixed $w \in \mathbb{R}^m_+$,

$$\text{minimize} \quad L(x, w) \text{ sub.to } x \in \mathbb{R}^n$$

$S_0 \subset \mathbb{R}^n$, $L(w, x) \leq f_0(x)$ if $x \in S_0$.

Hence $L^*(w) \equiv \min_{x \in \mathbb{R}^n} L(x, w) \leq \min_{x \in S_0} f_0(x) \ (\forall w \in \mathbb{R}^m_+)$

Inequality constrained optimization problem

$$\text{minimize} \quad f_0(x) \text{ sub.to } \; x \in S_0 = \{x \in \mathbb{R}^n : f_j(x) \leq 0 \; (j = 1, \ldots, m)\}$$

Lagrangian function:

$$L(x, w) \; = \; f_0(x) + w_1 f_1(x) + w_2 f_2(x) + \; \cdots \; + w_m f_m(x),$$

where $w \in \mathbb{R}^m_+ \equiv \{w = (w_1, w_2, \ldots, w_m) \in \mathbb{R}^m : w_j \geq 0\}$.

Example 2 (Polynomial optimization problem)

$$\begin{array}{ll} \text{minimize} & -x_1^3 + 2x_1x_2^2 \\ \text{sub.to} & x_1^4 + x_2^4 - 1 \leq 0, -x_1 \leq 0, -x_1^2 - x_2^2 - 0.5 \leq 0. \end{array}$$

$$\begin{aligned} L(x, w) \; \equiv \; & -x_1^3 + 2x_1x_2^2 + w_1(x_1^4 + x_2^4 - 1) \\ & + w_2(-x_1) + w_3(-x_1^2 - x_2^2 - 0.5) \\ = \; & w_1x_1^4 + w_1x_2^4 - x_1^3 + 2x_1x_2^2 \\ & - w_3x_1^2 - w_3x_2^2 - w_2x_1 - w_1 - 0.5w_3, \end{aligned}$$

where $w_1 \geq 0, w_2 \geq 0$.

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$\star$: Semidefinite Programming

Inequality constrained optimization problem

$$\text{minimize} \quad f_0(x) \text{ sub.to } \; x \in S_0 = \{x \in \mathbb{R}^n : f_j(x) \leq 0 \; (j = 1, \dots, m)\}$$

Lagrangian function:

$$L(x, w) \; = \; f_0(x) + w_1 f_1(x) + w_2 f_2(x) + \; \cdots \; + w_m f_m(x),$$

where $w \in \mathbb{R}^m_+ \equiv \{w = (w_1, w_2, \dots, w_m) \in \mathbb{R}^m : w_j \geq 0\}$.

Lagrangian relaxation problem: For every fix $w \in \mathbb{R}^m_+$,

$$\text{minimize } L(x, w) \text{ sub.to } \; x \in \mathbb{R}^n$$

Define $L^*(w) \equiv \min_{x \in \mathbb{R}^n} L(x, w) \leq \min_{x \in S_0} f_0(x) \; (\forall w \in \mathbb{R}^m_+)$

Lagrangian dual (The best Lagrangian relaxation problem)

$$\text{maximize }_{w \in \mathbb{R}^m_+} L^*(w)$$

$$\Updownarrow$$

$$\text{maximize }_{w \in \mathbb{R}^m_+} \text{minimize}_{x \in \mathbb{R}^n} L(x, w)$$

Inequality constrained optimization problem

$$\text{minimize} \quad f_0(x) \text{ sub.to } x \in S_0 = \{x \in \mathbb{R}^n : f_j(x) \leq 0 \ (j = 1, \ldots, m)\}$$

Example 2 (Polynomial optimization problem)

$$\begin{array}{ll} \text{minimize} & -x_1^3 + 2x_1x_2^2 \\ \text{sub.to} & x_1^4 + x_2^4 - 1 \leq 0, -x_1 \leq 0, -x_1^2 - x_2^2 - 0.5 \leq 0. \end{array}$$

$$\begin{aligned} L(x, w) \equiv & -x_1^3 + 2x_1x_2^2 + w_1(x_1^4 + x_2^4 - 1) \\ & + w_2(-x_1) + w_3(-x_1^2 - x_2^2 - 0.5) \\ = & \ w_1x_1^4 + w_1x_2^4 - x_1^3 + 2x_1x_2^2 \\ & - w_3x_1^2 - w_3x_2^2 - w_2x_1 - w_1 - 0.5w_3, \end{aligned}$$

where $w_1 \geq 0, w_2 \geq 0$.

Lagrangian dual: $\displaystyle\max_{(w_1, w_2) \geq 0} \ \min_{(x_1, x_2) \in \mathbb{R}^2} L(x, w)$.

- Although we introduce the Lagrangian dual, its minimization is difficult.

$\Rightarrow$ SOS, SDP relaxation

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$$\begin{array}{lll} \text{QOP} & \text{minimize} & f_0(x) \equiv x^T Q_0 x + q_0^T x \\ & \text{sub.to} & f_i(x) \equiv x^T Q_i x + q_i^T x + \pi_i \leq 0 \ (i = 1, \ldots, m). \end{array}$$

Here $x \in \mathbb{R}^n$: a vector variable,
$Q_i : n \times n$ symmetric matrix, $q_i \in \mathbb{R}^n$, $\pi_i \in \mathbb{R}$: constant

Notation: Given $n \times n$ symmetric matrix Q, X, $Q \bullet X = \sum_{j=1}^n \sum_{k=1}^n Q_{jk} X_{jk}$.

$$x^T Q x = \sum_{j=1}^n \sum_{k=1}^n Q_{jk} x_j x_k = Q \bullet x x^T.$$

Here xx^T becomes an $n \times n$ symmetric matrix;

$$xx^T = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} (x_1, x_2, \ldots, x_n) = \begin{pmatrix} x_1x_1 & x_1x_2 & \cdots & x_1x_n \\ x_2x_1 & x_2x_2 & \cdots & x_1x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_nx_1 & x_nx_2 & \cdots & x_nx_n \end{pmatrix}.$$

$$\text{QOP} \quad \text{minimize} \quad f_0(x) \equiv x^T Q_0 x + q_0^T$$
$$\text{sub.to} \quad f_i(x) \equiv x^T Q_i x + q_i^T x + \pi_i \leq 0 \ (i = 1, \ldots, m).$$

$\Updownarrow$ equivalent

$$\text{minimize} \quad Q_0 \bullet xx^T + q_0^T x$$
$$\text{sub.to} \quad Q_i \bullet xx^T + q_i^T x + \pi_i \leq 0 \ (i = 1, \ldots, m)$$

$\Updownarrow$ equivalent

$$\text{minimize} \quad Q_0 \bullet X + q_0^T x$$
$$\text{sub.to} \quad Q_i \bullet X + q_i^T x + \pi_i \leq 0 \ (i = 1, \ldots, m), \ X - xx^T = O$$

$\Downarrow$ SDP relaxation

$$\text{minimize} \quad Q_0 \bullet X + q_0^T x$$
$$\text{sub.to} \quad Q_i \bullet X + q_i^T x + \pi_i \leq 0 \ (i = 1, \ldots, m), \ X - xx^T \succeq O.$$

Here $A \succeq O \Leftrightarrow$ a symmetric matrix A is positive semidefinite, all eigen-values of A are nonnegative or $u^T A u \geq 0$ for $\forall u \in \mathbb{R}^n$.

$$\text{QOP} \quad \text{minimize} \quad f_0(x) \equiv x^T Q_0 x + q_0^T$$
$$\text{sub.to} \quad f_i(x) \equiv x^T Q_i x + q_i^T x + \pi_i \leq 0 \ (i = 1, \ldots, m).$$

$\Downarrow$ SDP relaxation

$$\text{minimize} \quad Q_0 \bullet X + q_0^T x$$
$$\text{sub.to} \quad Q_i \bullet X + q_i^T x + \pi_i \leq 0 \ (i = 1, \ldots, m), \ X - xx^T \succeq O.$$

$\Updownarrow$ equivalent $X - xx^T \succeq O \ \Leftrightarrow \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq O$

$$\text{SDP: minimize} \quad Q_0 \bullet X + q_0^T x$$
$$\text{sub.to} \quad Q_i \bullet X + q_i^T x + \pi_i \leq 0 \ (i = 1, \ldots, m), \ \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq O$$

- SDP is an extension of LP (Linear Program) to the space of symmetric matrices.
- SDPs with m, n = a few thousands can be solved by Interior-point methods, which was originally developed for LPs.

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Preparation — 1

$\boldsymbol{\lambda} : \mathbb{R}^n \rightarrow \mathbb{R}.$

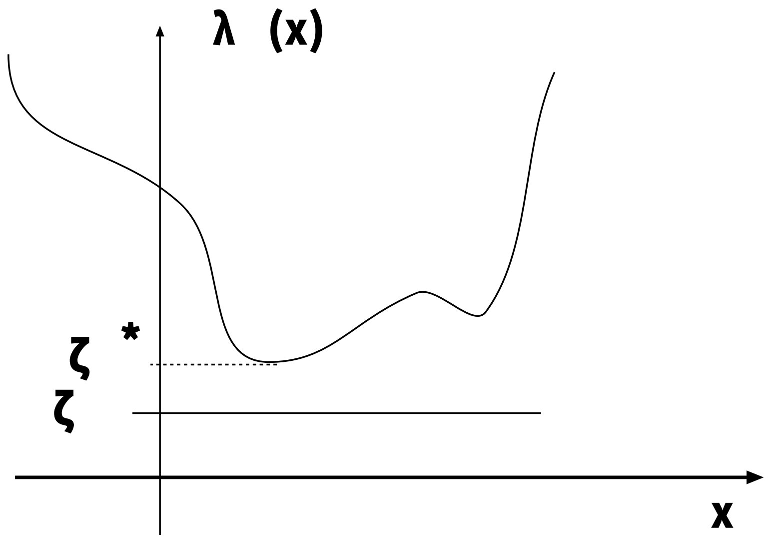
$$\zeta^* = \min_{\boldsymbol{x} \in \mathbb{R}^n} \boldsymbol{\lambda}(\boldsymbol{x})$$

$$\Updownarrow$$

Semi-infinite optimization problem (Optimization problem having an infinite number of inequality constraints)

$$\text{maximize } \zeta \text{ subject to } \boldsymbol{\lambda}(\boldsymbol{x}) - \zeta \geq 0 \ (\forall \boldsymbol{x} \in \mathbb{R}^n)$$

Here $\zeta \in \mathbb{R}$ denotes a variable but $\boldsymbol{x}$ an index parameter describing an infinite number of inequality constraints.



Preparation — 2

Nonnegative quadratic functions

$$\boldsymbol{\lambda}(\boldsymbol{x}) \equiv \boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x} + \boldsymbol{q}^T \boldsymbol{x} + \gamma \geq 0 \text{ for } \forall \boldsymbol{x} \in \mathbb{R}^n$$

$$\Updownarrow$$

$$\begin{aligned} \boldsymbol{\lambda}(\boldsymbol{x}) \ &: \ \text{a sum of squares of linear functions} \\ &= \sum_{i=1}^{k} \left(\boldsymbol{a}_i^T \boldsymbol{x} + \boldsymbol{b}_i\right)^2 \quad \text{for } \exists \boldsymbol{a}_i \in \mathbb{R}^n, \ \exists \boldsymbol{b}_i \in \mathbb{R}, \ \exists \boldsymbol{k} \in \mathbb{Z}_+. \end{aligned}$$

$$\Updownarrow$$

$$\boldsymbol{\lambda}(\boldsymbol{x}) \equiv \boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x} + \boldsymbol{q}^T \boldsymbol{x} + \gamma = (1, \boldsymbol{x}^T) \boldsymbol{V} \begin{pmatrix} 1 \\ \boldsymbol{x} \end{pmatrix} \text{ for } \exists \boldsymbol{V} \succeq \boldsymbol{O} \ \text{ and } \forall \boldsymbol{x} \in \mathbb{R}^n$$

Preparation — 3

Nonnegative polynomial functions with degree $\ell \le 2m$.

$$\lambda(x) \ge 0 \text{ for } \forall x \in \mathbb{R}^n$$

$$\Uparrow$$

$$\lambda(x) \;:\; \text{a sum of squares of polynomial functions with degree } \le m$$

$$= \sum_{i=1}^{k} g_i(x)^2$$

for $\exists$polynomial functions $g_i(x)$ with degree $\le m$, $\exists k \in \mathbb{Z}_+$.

$$\Updownarrow$$

$$\lambda(x) = u(x) V u(x)^T \text{ for } \exists V \succeq O \text{ and } \forall x \in \mathbb{R}^n,$$

where $u(x) = (1, x_1, \ldots, x_n, x_1^2, x_1x_2, \ldots, x_n^2, \ldots, x_1^m, \ldots, x_n^m)$

(a row vector of basis for a real valued polynomial of degree m)

Preparation — 4

Example. Characterization of a nonnegative quadratic function $\lambda(x) = d + bx_1 + cx_2 + x_1^2 + ax_1x_2 + 2x_2^2$: Choose a, b, c, d such that $\lambda(x) \geq 0$ for $\forall x \in \mathbb{R}^2$

$$d + bx_1 + cx_2 + x_1^2 + ax_1x_2 + 2x_2^2 = (1, x_1, x_2)V \begin{pmatrix} 1 \\ x_1 \\ x_2 \end{pmatrix}$$

$$= V_{00} + 2V_{01}x_1 + 2V_{02}x_2 + V_{11}x_1^2 + 2V_{12}x_1x_2 + V_{22}x_2^2$$

$$\text{for } \exists V = \begin{pmatrix} V_{00} & V_{01} & V_{02} \\ V_{01} & V_{11} & V_{12} \\ V_{02} & V_{12} & V_{22} \end{pmatrix} \succeq O$$

$\Updownarrow$ The coefficients of $x_1, x_2, x_1x_2, x_1^2, x_2^2$ in both side must coincide to each other, respectively.

$$d = V_{00},\ b = 2V_{01},\ c = 2V_{02},\ 1 = V_{11},\ a = 2V_{12},\ 2 = 2V_{22}, V \succeq O$$

(Linear Matrix Inequality)

$$\text{QOP} \quad \text{minimize} \quad f_0(x) \equiv x^T Q_0 x + q_0^T$$
$$\text{sub.to} \quad f_i(x) \equiv x^T Q_i x + q_i^T x + \pi_i \leq 0 \ (i = 1, \ldots, m).$$

Lagrangian relaxation with a fixed parameter $w \in \mathbb{R}^m_+$

$$\text{minimize} \quad L(x, w) \equiv f_0(x) + \sum_{i=1}^{m} w_i f_i(x) \text{sub.to} \quad x \in \mathbb{R}^n$$

⇕ equivalent

$$\text{maximize } \zeta \text{ sub.to} \quad f_0(x) + \sum_{i=1}^{m} w_i f_i(x) - \zeta \geq 0 \ (\forall x \in \mathbb{R}^n)$$

⇕ equivalent

$$\text{maximize } \zeta \text{ sub.to} \quad f_0(x) + \sum_{i=1}^{m} w_i f_i(x) - \zeta = (1, x^T) V \begin{pmatrix} 1 \\ x \end{pmatrix} \quad \text{for} \exists V \succeq O.$$

⇕ Comparison of coefficients of every monomial in both side

$$\text{SDP:} \quad \text{maximize } \zeta$$
$$\text{sub.to Linear equations in } V, V \succeq O$$

$$\text{QOP} \quad \text{minimize} \quad f_0(x) \equiv x^T Q_0 x + q_0^T$$
$$\text{sub.to} \quad f_i(x) \equiv x^T Q_i x + q_i^T x + \pi_i \leq 0 \ (i = 1, \ldots, m).$$

Lagrangian relaxation with a fixed parameter $w \in \mathbb{R}^m_+$

$\Updownarrow$ equivalent

SDP: maximize ζ
sub.to Linear equations in $V, V \succeq O$

maximization in $w \in \mathbb{R}^n_+$ $\Downarrow$ The best Lagrangian relaxation

SDP: maximize ζ
sub.to Linear equations in $w \in \mathbb{R}^m_+$ and V, $V \succeq O$

SDP relaxation of QOP $\Updownarrow$ dual

$$\text{minimize} \quad Q_0 \bullet X + q_0^T x$$
$$\text{sub.to} \quad Q_i \bullet X + q_i^T x + \pi_i \leq 0 \ (i = 1, \ldots, m), \ \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq O$$

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$$\text{QOP} \quad \text{minimize} \quad f_0(x) \equiv x^T Q_0 x + q_0^T \quad \text{sub.to} \quad f_i(x) \equiv x^T Q_i x + q_i^T x + \pi_i \le 0 \ (i = 1, \ldots, m).$$

QOP	$\Rightarrow$	SDP relaxation of QOP
$\Downarrow$		$\Updownarrow$ Duality theory
Lagrangian dual of QOP	$\Leftrightarrow$	Dual SDP relaxation of QOP

- $\Leftrightarrow$ follows from

  Nonnegative quadratic functions = Sum of squares of linear functions

- Optimal values

  QOP $\geq$ Lagrangian dual = SDP = Dual SDP.

- Computation

  SDP, Dual SDP can be solved by interior-point methods.

POP minimize $f_0(x)$ sub.to $f_i(x) \leq 0$ $(i = 1, \ldots, m)$,
where $f_i(x)$ denotes a polynomial in $x \in \mathbb{R}^n$ $(i = 0, 1, 2, \ldots, m)$.

POP	$\Rightarrow$	**SDP relaxation of POP**
$\Downarrow$		$\Updownarrow$ **Duality theory**
Lagrangian dual of POP	$\Rightarrow$	**Dual SDP relaxation of POP**

- $\Rightarrow$ **follows from**

  Nonnegative polynomials $\supset$ Sum of squares of polynomials

- **Optimal values**

  QOP $\geq$ Lagrangian dual $\geq$ SDP $=$ Dual SDP.

- **Computation**

  SDP, Dual SDP can be solved by interior-point methods.

This presentation material is available at

http://www.is.titech.ac.jp/∼kojima/talk.html

Thank you!

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