2010/07/09

2010年度前学期数学演習３類（Ｐクラス）第１２回演習問題

注: 7月23日は、演習の微分積分の期末テストを行います。

1. 教科書類の持込禁止
2. 試験は演習を行っている教室で行う (13時20分には着席のこと)
3. 問題 (2-1) と $\tan^{-1}(x)$ の Maclaurin 展開，偏導関数は必ず出すのでよく復習しておくこと
4. 他に、導関数, 関数の連続性，ロピタルの定理，原点における極限値，極値も重要な項目である
5. 質問がある場合には、数学相談室や山下まで相談に来てください (先に Makoto.Yamashita@is.titech.ac.jp にメールをください)

(12-1) この問題を解答用紙に解答すること
次の関数 $f(x,y)$ の1次,2次の偏導関数をすべて求めよ。[Hint: 教科書 P99]

$$\begin{aligned} &\text{(i)} && f(x,y) = e^{xy} \\ &\text{(ii)} && f(x,y) = \log(x^2+y^2) \\ &\text{(iii)} && f(x,y) = \tan^{-1}\left(\frac{y}{x}\right) \end{aligned}$$

(12-2) 次の関数 $f(x,y)$ の極値を求めよ。ただし、a は $a>0$ となる定数である。[Hint: 教科書 P113 の例1]

$$\begin{aligned} &\text{(i)} && f(x,y) = xy(x^2+y^2-1) \\ &\text{(ii)} && f(x,y) = xy + \frac{a}{x} + \frac{a}{y} \end{aligned}$$

(12-3) $u = \frac{1}{\sqrt{x^2+y^2+z^2}}$ のとき、$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$ を求めよ。

(12-4) $u = f(x,y)$ は C^2 級関数とする。つまり、2階微分も連続であるため、$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$ である。
$x = r\cos\theta, y = r\sin\theta$ とするとき、

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

となることを示せ。
Hint: 連鎖公式より

$$\begin{aligned}
&\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}, \qquad \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}, \\
&\frac{\partial^2 u}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial r}\right) + \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}\right) \\
= \quad &\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right)\frac{\partial x}{\partial r} + \frac{\partial u}{\partial x}\frac{\partial}{\partial r}\left(\frac{\partial x}{\partial r}\right) + \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial y}\right)\frac{\partial y}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial}{\partial r}\left(\frac{\partial y}{\partial r}\right) \\
= \quad &\left\{\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)\frac{\partial y}{\partial r}\right\}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial x}\frac{\partial}{\partial r}\left(\frac{\partial x}{\partial r}\right) \\
&+\left\{\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)\frac{\partial y}{\partial r}\right\}\frac{\partial y}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial}{\partial r}\left(\frac{\partial y}{\partial r}\right) \\
= \quad &\frac{\partial^2 u}{\partial x^2}\left(\frac{\partial x}{\partial r}\right)^2 + 2\frac{\partial^2 u}{\partial y \partial x}\left(\frac{\partial x}{\partial r}\right)\left(\frac{\partial y}{\partial r}\right) + \frac{\partial^2 u}{\partial y^2}\left(\frac{\partial y}{\partial r}\right)^2 + \frac{\partial u}{\partial x}\left(\frac{\partial^2 x}{\partial r^2}\right) + \frac{\partial u}{\partial y}\left(\frac{\partial^2 y}{\partial r^2}\right)
\end{aligned}$$

略解

(12-1) (i) $\frac{\partial f}{\partial x}=ye^{xy}$, $\frac{\partial f}{\partial y}=xe^{xy}$, $\frac{\partial^2 f}{\partial x^2}=y^2e^{xy}$, $\frac{\partial^2 f}{\partial y^2}=x^2e^{xy}$, $\frac{\partial^2 f}{\partial x\partial y}=(1+xy)e^{xy}$, $\frac{\partial^2 f}{\partial y\partial x}=(1+xy)e^{xy}$.
(ii) $\frac{\partial f}{\partial x}=\frac{2x}{x^2+y^2}$, $\frac{\partial f}{\partial y}=\frac{2y}{x^2+y^2}$, $\frac{\partial^2 f}{\partial x^2}=\frac{-2x^2+2y^2}{(x^2+y^2)^2}$, $\frac{\partial^2 f}{\partial y^2}=\frac{2x^2-2y^2}{(x^2+y^2)^2}$, $\frac{\partial^2 f}{\partial x\partial y}=\frac{-4xy}{(x^2+y^2)^2}$, $\frac{\partial^2 f}{\partial y\partial x}=\frac{-4xy}{(x^2+y^2)^2}$
(iii) $\frac{\partial f}{\partial x}=\frac{-y}{x^2+y^2}$, $\frac{\partial f}{\partial y}=\frac{x}{x^2+y^2}$, $\frac{\partial^2 f}{\partial x^2}=\frac{2xy}{(x^2+y^2)^2}$, $\frac{\partial^2 f}{\partial y^2}=\frac{-2xy}{(x^2+y^2)^2}$, $\frac{\partial^2 f}{\partial x\partial y}=\frac{-x^2+y^2}{(x^2+y^2)^2}$, $\frac{\partial^2 f}{\partial y\partial x}=\frac{-x^2+y^2}{(x^2+y^2)^2}$

(12-2) (i) $(x,y)=\left(\pm\frac{1}{2},\pm\frac{1}{2}\right)$ で極小値 $-\frac{1}{8}$, $(x,y)=\left(\pm\frac{1}{2},\mp\frac{1}{2}\right)$ で極大値 $\frac{1}{8}$
(ii) $(x,y)=(\sqrt[3]{a},\sqrt[3]{a})$ で極小値 $3\sqrt[3]{a}$

(12-3) 0 となる。

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}\left(\frac{-3x^2}{x^2+y^2+z^2}+1\right)\\
\frac{\partial^2 u}{\partial y^2} &= \frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}\left(\frac{-3y^2}{x^2+y^2+z^2}+1\right)\\
\frac{\partial^2 u}{\partial z^2} &= \frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}\left(\frac{-3z^2}{x^2+y^2+z^2}+1\right)
\end{aligned}$$

(12-4) 教科書 P106 にあるとおり、以下の式を用いる。

$$\begin{aligned}
\frac{\partial u}{\partial r} &= \cos\theta\frac{\partial u}{\partial x}+\sin\theta\frac{\partial u}{\partial y}\\
\frac{\partial u}{\partial \theta} &= -r\sin\theta\frac{\partial u}{\partial x}+r\cos\theta\frac{\partial u}{\partial y}\\
\frac{\partial^2 u}{\partial r^2} &= \cos^2\theta\frac{\partial^2 u}{\partial x^2}+2\sin\theta\cos\theta\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+\sin^2\theta\frac{\partial^2 u}{\partial y^2}\\
\frac{\partial^2 u}{\partial \theta^2} &= -r\cos\theta\frac{\partial u}{\partial x}-r\sin\theta\frac{\partial u}{\partial y}+r^2\sin^2\theta\frac{\partial^2 u}{\partial x^2}-2r^2\sin\theta\cos\theta\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+r^2\cos^2\theta\frac{\partial^2 u}{\partial y^2}
\end{aligned}$$